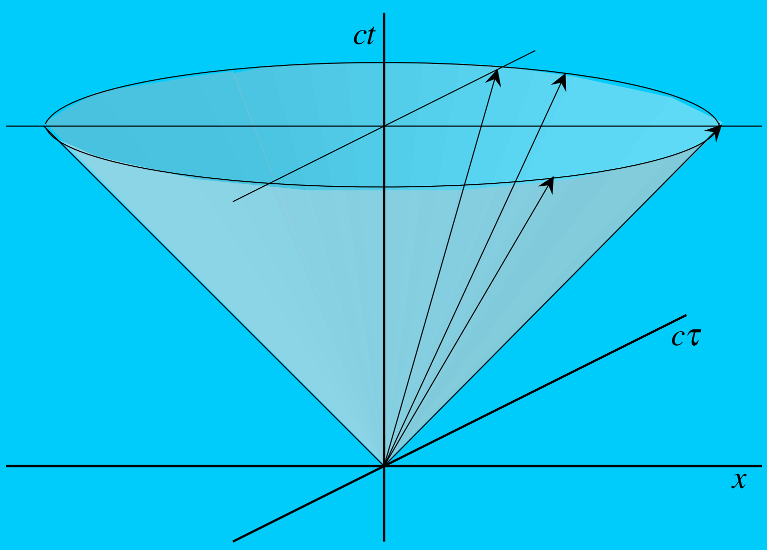
Proper Time as Fourth Coordinate



Hans Montanus

Proper Time as Fourth Dimension

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Preface

The theory of relativity is part of physics for more than a century. It delivered predictions of several observed phenomena. One could say that the theory of relativity is a very successful model. The predictions are so convincing that one is easily inclined to regard the theory of relativity as an established fact like a theorem in mathematics. However, the theory of relativity still is a model. In this book we consider an alternative model where the proper time of an object is taken as its fourth coordinate. In the new model spacetime is Euclidean and there is a preferred frame of reference. The new model might seem quite opposite to the special theory of relativity which is based on a relative Minkowski spacetime. Despite the differences the predictions in the two spacetime models are comparable. For many phenomena the predictions are even identical. A spacetime with a preferred frame of reference will be denoted as an absolute Euclidean spacetime. The word ‘absolute’ solely refers to the preferred frame. It does not mean that all clocks tick equally fast.

An introduction will be given of the concept of an absolute Euclidean spacetime. Famous relativity experiments will be considered in an absolute Euclidean spacetime. The kinematics in an absolute Euclidean spacetime will be compared with the relative Minkowski spacetime. Similarities and differences with the special theory of relativity will be illuminated. Also dynamics in an absolute Euclidean will be considered. In particular much attention will be paid to gravity. For this, we do not need the concept of curvature. Gravity will be described in a flat absolute Euclidean spacetime. Explicit calculations will be given for the bending or precession of orbital motions in the vicinity of a spherical source mass, a bipole mass, a disk and an oblate spheroid. Electrodynamics will be considered from tensor point of view and from the geometric algebra point of view. At the end attention will be paid to some various topics such as intrinsic redshift and the arrow of time.

september 23, 2023, Hans Montanus

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Chapter 1

Absolute Euclidean spacetime

1.1 Introduction

To determine the positions of particles an observer has a yardstick at his disposal. By means of a yardstick a three dimensional orthogonal coordinate frame is created. The coordinates are usually denoted as (x, y, z). In physics particles move in time. To follow the times at which a moving particle is passing subsequent positions the observer has a clock at his disposal. The time of this clock will be denoted as t. It might seem natural to put the time values delivered by the clock of the observer on a different axis. In doing so, the observer takes, at least mathematically, the time of his clock as a fourth dimension. In figure 1.1 a (x, t) diagram is shown with three inertial particles moving of simultaneously from the origin of the coordinate frame with velocity $0.6c$, $0.8c$ and c respectively.

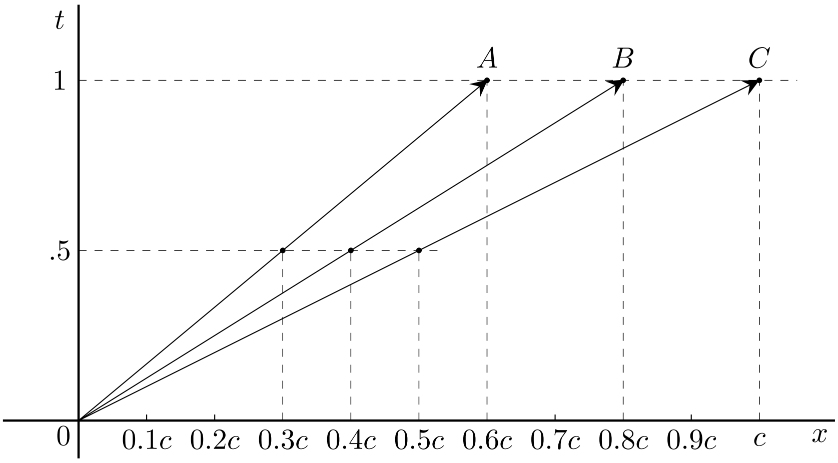

Figure 1.1: Diagram of the coordinate x and the time t for three moving objects A, B and C during one second.

For particle A we have after 0.5 seconds $(x, t) = (0.3c, 0.5)$ and after 1 second $(x, t) = (0.6c, 1)$. For particle B this is $(x, t) = (0.4c, 0.5)$ and after 1 second $(x, t) = (0.8c, 1)$, and for particle C

this is $(x,t) = (0.5c, 0.5)$ and after 1 second $(x,t) = (c, 1)$. In figure 1.1 it is not clear if t is a coordinate or a parameter which keeps track the order of events. Classically t is a parameter for the order of events in a three dimensional space: $\big(x(t), y(t), z(t)\big)$. Since the theory of relativity t is regarded as a coordinate. To give it a dimension of length it is multiplied by the velocity of light c: $\big(x, y, z, ct\big)$. With this provision figure 1.1 turns into figure 1.2.

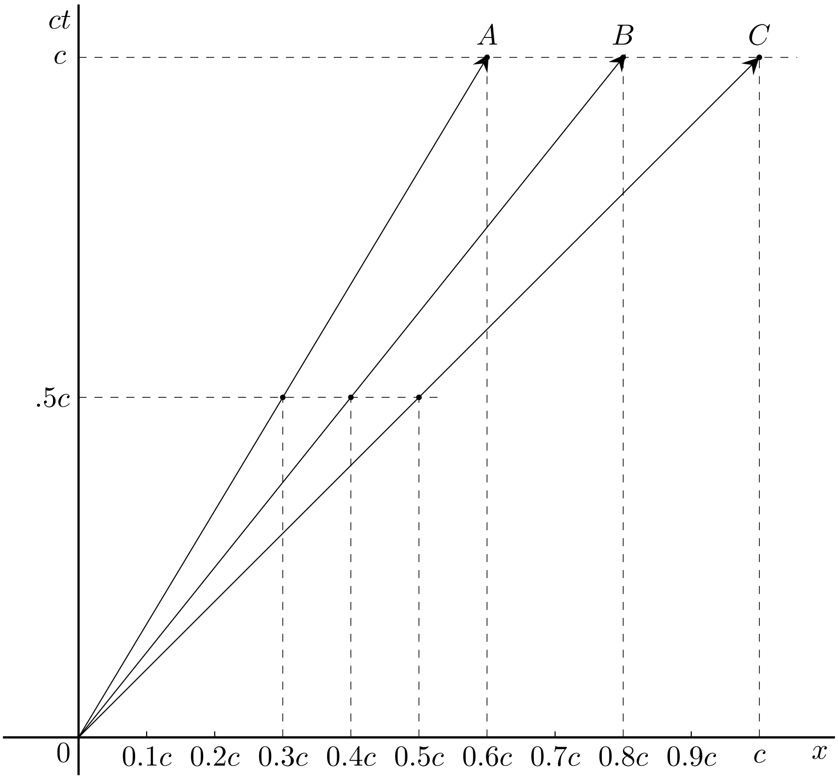

Figure 1.2: Diagram of the coordinates x and ct for moving objects A, B and C during one second.

If ct is a coordinate then one would expect that at an instant of time the ct coordinates of different objects can take on different values in the same way as the x coordinates of different objects can take on different values. This is not the case. It only seems natural for the ct coordinates of different objects to be equal at each instant of time if one assumes the coordinates to be related to the instant of time. That is, it only seems natural for the ct coordinates of different objects to be equal at each instant of time if t also is the parameter which determines the order of events. Indeed for $\big(x(t), y(t), z(t), ct(t)\big)$ the coordinate ct will have a value c times t at each instant of parameter time t. It would imply, however, that t simultaneously acts as a fourth coordinate and as a parameter. To avoid the ambiguity one should either take ct as a fourth coordinate and look for another parameter than t, or one should take t as the parameter and look for another coordinate then ct.

Suppose a clock would be attached to each of the moving particles A, B and C. As we know from relativity theory and experiments a clock runs slower the faster it moves. The imaginary attached clock is known as the proper time τ. In figure 1.3 the proper times of the particles A, B and C after 0.25, 0.5, 0.75 and 1 second have been attached to the trajectories of A, B and C.

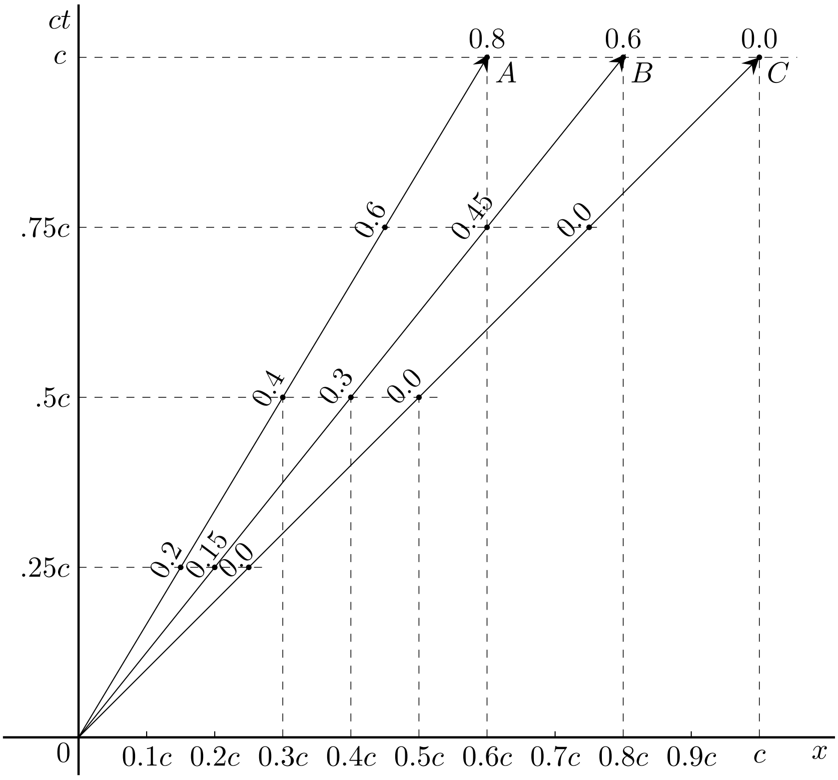

Figure 1.3: Diagram of the coordinates x and ct for moving objects A, B and C during one second. The individual proper times are attached to each trajectory.

The proper time can fulfil the role of either the missing parameter or of the missing fourth coordinate. That is, one can either take ct as the fourth coordinate and τ as the parameter, or one can take $c\tau$ as the fourth coordinate and t as the parameter.

The first path is followed in the theory of relativity: $\big(x(\tau), y(\tau), z(\tau), ct(\tau)\big)$. As a consequence, in relativity theory each particle will have its own parameter since each particle has its own proper time. As another consequence, in relativity theory the fourth coordinates of different particles is a single value at each instant of time.

The second path will be followed in this book: $\big(x(t), y(t), z(t), c\tau(t)\big)$. As a consequence, in the present analysis each particle will have its own fourth coordinates since each particle has its own proper time. As another consequence, in the present analysis all particles are parameterised by a single parameter.

To illuminate the second path we cast the motion of the particles A, B and C in a $(x, c\tau)$ diagram, see figure 1.4.

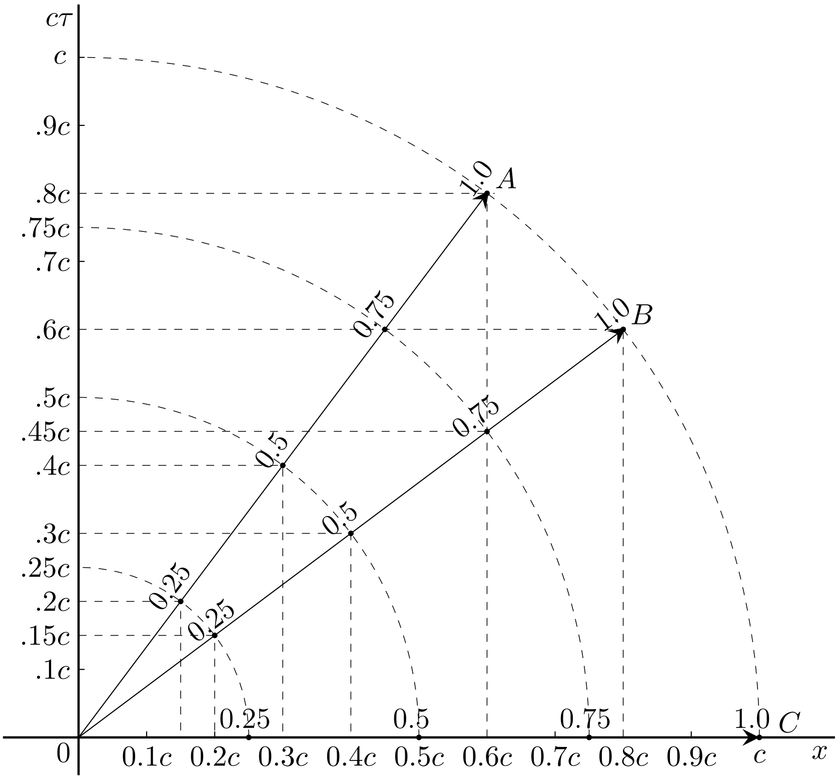

Figure 1.4: Diagram of the coordinates x and $c\tau$ for moving objects A, B and C during one second. The parameter times are attached to each trajectory.

We see how a single time t parameterises the motion of all the moving particles. We also see how different events have different fourth coordinates even if they occur at the same parameter time. For particles which moved of from the origin of the $(x, c\tau)$ diagram the $(x(t), c\tau(t))$ coordinates are positioned on a circle with radius ct. At parameter time $t = 0.25$ seconds particle A is at $(x_A, c\tau_A) = (0.15c, 0.2c)$. At each instant of time t particle A is at $(x_A, c\tau_A) = (0.6ct, 0.8ct)$. For particle B and C this is $(x_B, c\tau_B) = (0.8ct, 0.6ct)$ and $(x_C, c\tau_C) = (ct, 0)$ respectively. For all three particles there holds $x^2 + c^2\tau^2 = c^2t^2$. Since $x = vt$ we get $c^2\tau^2 = t^2(c^2 - v^2)$. The latter illuminates the Pythagorean origin of the relation $\tau = t\sqrt{1 - v^2/c^2}$. This is the basic concept of time of the present analysis in a nutshell.

1.2 Minkowski metric.

Important ingredients of special relativity theory [1] are the Minkowski metric and the Lorentz transformations. Special relativity theory (SRT) allows for an explanation of the Fizeau experiment, the Compton effect, pair-annihilation, etc. Without doubt the SRT predicts relativistic phenomena very well. Despite these successes we ask ourselves whether relativistic phenomena can also be predicted by an alternative spacetime model. To this end we consider some consequences of the Minkowski metric.

We start with two objects, A and B, moving with constant velocities in the x-direction and passing the origin at $t = 0$. The corresponding Minkowski diagram is as shown in figure 1.5.

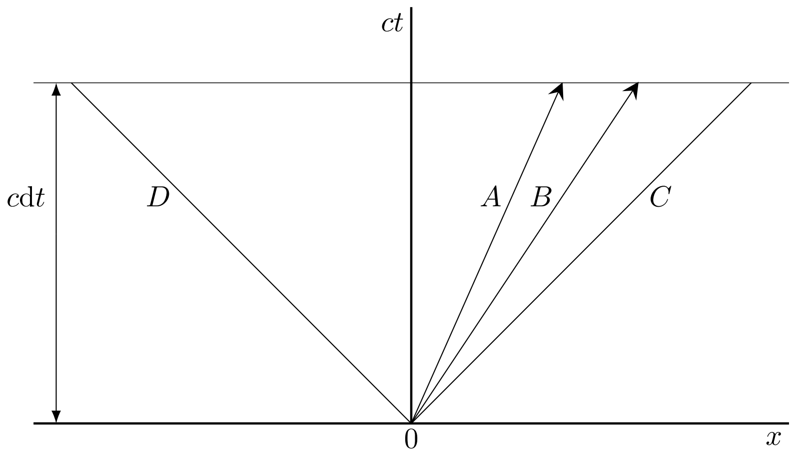

Figure 1.5: Minkowski diagram for two moving objects A and B during an infinitesimal time lapse $\mathrm{d}t$. The lines C and D represent the light cone.

According to Minkowski's implementation[2] the distance of the infinitesimal displacement of object A is defined by both

$$\mathrm{d}s_A^2 = c^2\mathrm{d}t_A^2 - \mathrm{d}x_A^2 \tag{1.1}$$

and

$$\mathrm{d}s_A^2 = c^2\mathrm{d}\tau_A^2 \,. \tag{1.2}$$

The first is referred to as the metric distance and the second as the proper time distance. Elimination of $\mathrm{d}s_A$ leads to

$$c^2\mathrm{d}\tau_A^2 = c^2\mathrm{d}t_A^2 - \mathrm{d}x_A^2 \,. \tag{1.3}$$

The latter is identical to the relativistic time dilatation:

$$\mathrm{d}\tau_A = \mathrm{d}t_A\sqrt{1 - v_A^2/c^2} \,, \tag{1.4}$$

where $v_A = \mathrm{d}x_A/\mathrm{t}_A$ is the velocity of object A.

For a consistent spacetime model the metric distance between two objects should not run into conflict with the proper time distance between two objects. In line with equation (1.1) the metric distance between A and B at time Δt is

$$\Delta s_{AB}^2 = c^2\Delta t_{AB}^2 - \Delta x_{AB}^2 \,, \tag{1.5}$$

where $\Delta t_{AB} = t_A - t_B$ and $\Delta x_{AB} = x_A - x_B$. Since t_A equals t_B at any instant of time, it is reduced to

$$\Delta s_{AB}^2 = -\left(x_A - x_B\right)^2 \; . \tag{1.6}$$

In line with equation (1.2) the proper time distance between A and B at time Δt is

$$\Delta s_{AB}^2 = c^2 \Delta\tau_{AB}^2 \, , \tag{1.7}$$

where $\Delta\tau_{AB} = \tau_A - \tau_B$. The equations (1.6) and (1.7) run into conflict since $-\left(x_A - x_B\right)^2 \leq 0$ while $c^2\left(\tau_A - \tau_B\right)^2 \geq 0$. One often uses the $(-,+,+,+)$ metric in stead of the $(+,-,-,-)$ metric. If we apply the $(-,+,+,+)$ metric the equation (1.5), equation (1.6) and equation (1.7) read

$$\Delta s_{AB}^2 = -c^2 \Delta t_{AB}^2 + \Delta x_{AB}^2 \, , \tag{1.8}$$

$$\Delta s_{AB}^2 = \left(x_A - x_B\right)^2 \tag{1.9}$$

and

$$\Delta s_{AB}^2 = -c^2 \Delta\tau_{AB}^2 \tag{1.10}$$

respectively. In this case there is still a conflict: $\left(x_A - x_B\right)^2 \geq 0$ while $-c^2\left(\tau_A - \tau_B\right)^2 \leq 0$. In other words, for both the $(-,+,+,+)$ and $(+,-,-,-)$ metric either the metric distance or the proper time distance is imaginary. And in either case the metric distance and proper time distance have opposite sign.

There is another odd consequence of Minkowski's implementation. As mentioned before, for simultaneous positions of two objects A and B we have $t_A = t_B$. Equation (1.5) or equation (1.8) then reduce to equation (1.6) and equation (1.9) respectively. That is, the simultaneous distances between objects are purely spatial. Alternatively, in a two, three or four dimensional Minkowski diagram simultaneous events always lie on a horizontal line, plane or hyperplane respectively. However, at a given instant of time a true fourth coordinate should have the freedom to take on a value different from that instant of time. Simultaneity is indissolubly connected with the time ordering of events. The time ordering of events should be determined by a time parameter and not by a fourth coordinate. According to SRT, for an ensemble of objects, A, B, C ..., the proper times, τ_A, τ_B, τ_C ..., act as a parameter for the motions of A, B, C ... For each object the motion can be parameterised by the proper time of the object. However, none of the proper times, τ_A, τ_B, τ_C ..., is fit to determine the order of events in an ensemble of motions. None of the proper times, τ_A, τ_B, τ_C ..., can be used for an unambiguous determination of simultaneity. A consistent parameterisation requires a single parameter for the determination of simultaneity and the order of events. A consistent metric requires a fourth coordinate which differs for different simultaneous events and which can take on any value at each instant of the parameter time.

1.3 Euclidean metric

A consistent parameterisation and a consistent metric is achieved in a four dimensional space-time by means of a new concept of time [3–6] . According to the new concept, the proper times of objects are taken as their fourth coordinates (time coordinates), while the clock of an observer is used to keep track of the order of events. That is, the proper time of an observer is taken as the evolution parameter. At each instant of parameter time all the four coordinates may differ from one event to another, even for simultaneous events. According to the new concept of time distances will be measured by the Euclidean metric:

$$s^2 = x^2 + y^2 + z^2 + c^2\tau^2 \,. \tag{1.11}$$

For the infinitesimal displacement of a single object A this is

$$\mathrm{d}s_A^2 = \mathrm{d}x_A^2 + \mathrm{d}y_A^2 + \mathrm{d}z_A^2 + c^2\mathrm{d}\tau_A^2 \,. \tag{1.12}$$

For the Euclidean distance between two events, A and B, this is

$$\Delta s_{AB}^2 = \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2 + c^2\Delta\tau_{AB}^2 \,, \tag{1.13}$$

where $\Delta\tau_{AB} = \tau_A - \tau_B$ and $\Delta x_{AB} = x_A - x_B$, $\Delta y_{AB} = y_A - y_B$ and $\Delta z_{AB} = z_A - z_B$. Notice that $\Delta\tau_{AB}$ is, in general, not zero. When two objects A and B with different history collide, their clocks will, in general, read different at the moment of collision. Although $\Delta x_{AB} = \Delta y_{AB} = \Delta z_{AB} = 0$ at the moment of collision, there might be a non zero distance $\Delta s_{AB} = c\,(\tau_A - \tau_B)$ at the moment of collision. The different proper times at the moment of collision is harmless for the physics of the collision.

In analogy with equation (1.12) the infinitesimal displacement of an observer O is

$$\mathrm{d}s_O^2 = \mathrm{d}x_O^2 + \mathrm{d}y_O^2 + \mathrm{d}z_O^2 + c^2\mathrm{d}\tau_O^2 \,. \tag{1.14}$$

Since the displacement of the observer with respect to his own reference frame is zero, we have $\mathrm{d}x_O = \mathrm{d}y_O = \mathrm{d}z_O = 0$. And since the proper time of the observer is used as parameter time we write $\mathrm{d}\tau_O = \mathrm{d}t$. Hence,

$$\mathrm{d}s_O^2 = c^2\mathrm{d}t^2 \,. \tag{1.15}$$

In analogy to the situation in the SRT the following principle will be postulated: in the absence of gravitation and other potential energies all objects move with equal four dimensional Euclidean velocities. Explicitly, the postulate reads

$$\mathrm{d}s_O^2 = \mathrm{d}s_A^2 \,. \tag{1.16}$$

As a consequence equation (1.12) can be written as

$$c^2\mathrm{d}t^2 = \mathrm{d}x_A^2 + \mathrm{d}y_A^2 + \mathrm{d}z_A^2 + c^2\mathrm{d}\tau_A^2 \,. \tag{1.17}$$

If we also leave the index of the object, it reads

$$c^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + c^2\dot{\tau}^2 \,, \tag{1.18}$$

where the dot stands for the derivative with respect to the time parameter t. The latter equation expresses the following: in a Euclidean spacetime everything moves with a four dimensional velocity equal to the speed of light.

To show the connection with SRT we rearrange the equation (1.17) to

$$-\, c^2 \mathrm{d}\tau_A^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}x_A^2 + \mathrm{d}y_A^2 + \mathrm{d}z_A^2 \,. \tag{1.19}$$

This is just the four dimensional analogue of equation (1.3) known from SRT. In this respect we do not diverge from SRT. The difference is only in the interpretation: we take the proper time of an observer as a parameter time and not as the fourth coordinate, and we take the proper time of an object (or an event) as the fourth coordinate and not as a parameter time. Newburgh and Phipps were probably the first who advocated the alternative concept of a Euclidean metric with proper time as a fourth coordinate [7,8] .

1.4 Euclidean diagram

According to equation (1.17) and equation (1.18) everything moves with four dimensional velocity c with respect to the frame of an observer. This implies that objects moving to the origin of the observers frame all reach a circle with radius $c\mathrm{d}t$ after an infinitesimal increment $\mathrm{d}t$ of parameter time. The situation is illustrated in figure 1.6.

In the Euclidean diagram of figure 1.6 the sign of the proper times of objects A and B are positive and negative respectively. We will therefore regard them as a particle and antiparticle respectively.

We will determine whether the Euclidean spacetime is relative or absolute. With absolute we mean that there is a preferred frame of reference. With relative we mean that the frame of some observer is as good as any other. For observer O the situation is illustrated in figure 1.7.

The frame of observer O is denoted as $c\tau_O$ for the time axis and x_O for the x-axis. Object A is moving with respect to the frame of O with spatial speed $0.6c$ in the x direction. Its proper time speed therefore is $0.8c$. The total speed of object A with respect to the frame of O is $\sqrt{0.6^2c^2 + 0.8^2c^2} = c$. Object B is moving with respect to the frame of O with spatial speed $0.8c$ in the x direction. Since its proper time is negative, its proper time speed is $-0.6c$. The total speed of object B with respect to O is $\sqrt{0.8^2c^2 + (-0.6)^2c^2} = c$.

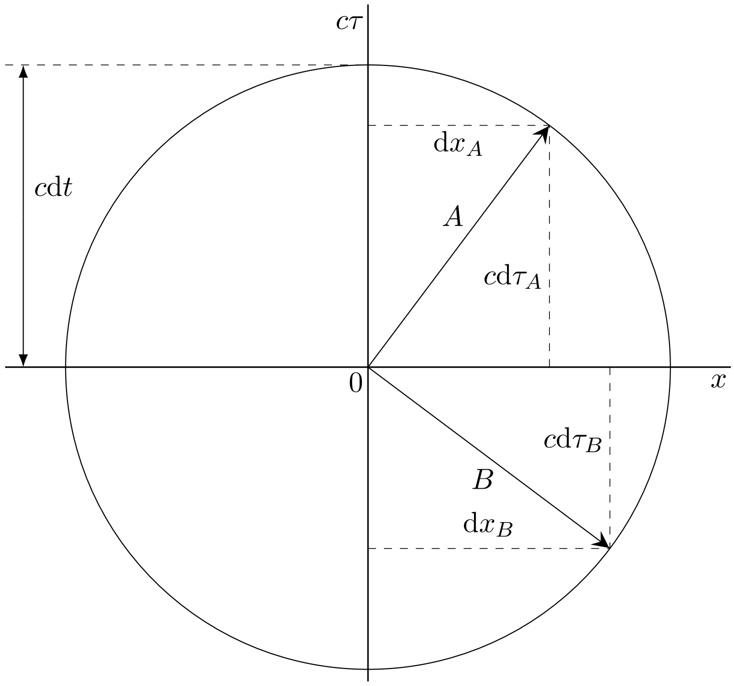

Figure 1.6: Euclidean diagram for two moving objects A and B during an infinitesimal time lapse dt. The proper times of objects A and B have opposite proper times.

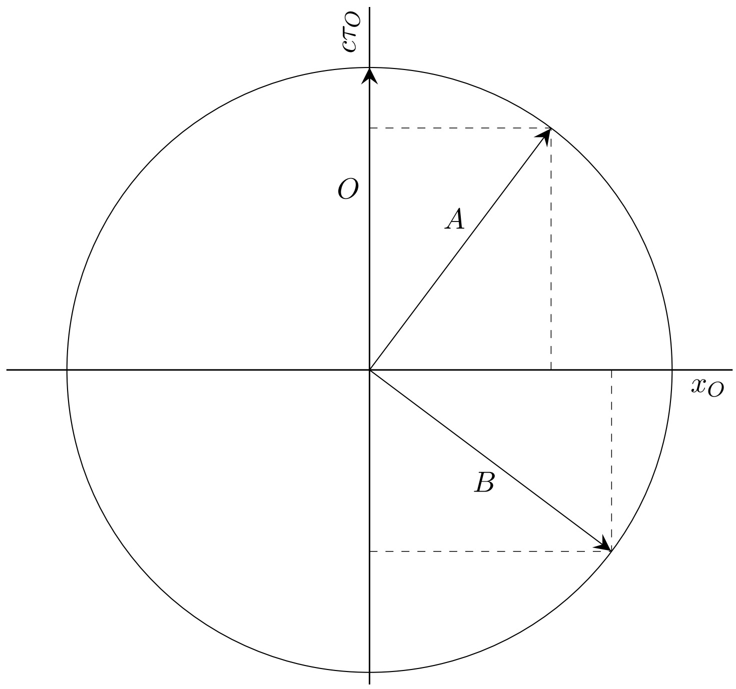

Figure 1.7: Two different objects, A and B as observed by observer O.

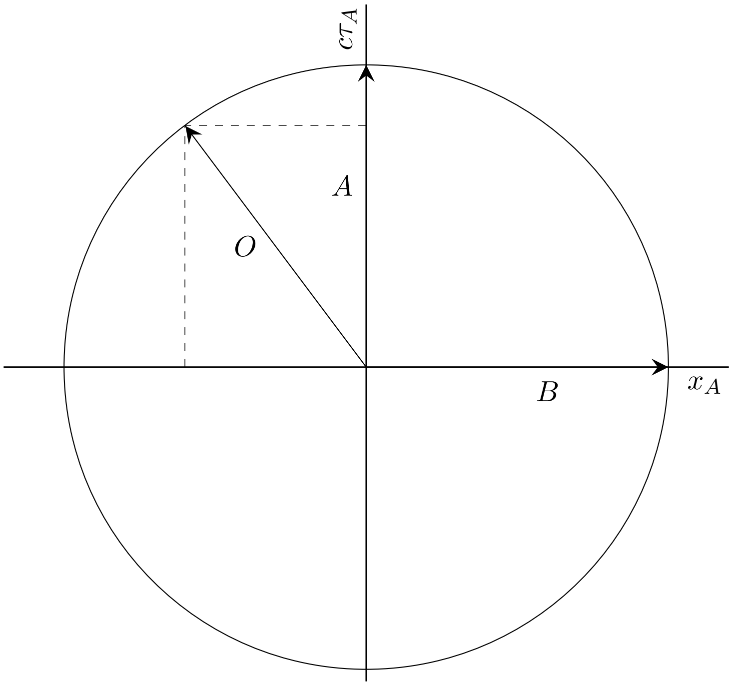

Figure 1.8: Two different objects, O and B as observed by observer A in a relative Euclidean spacetime.

Now we regard object A as an observer. For observer A the situation is shown in figure 1.8. The frame of observer A is denoted as $c\tau_A$ for the time axis and x_A for the x-axis. In a relative spacetime the object O is moving with respect to the frame of A with spatial speed $-0.6c$ in the x direction. Its proper time speed therefore is $0.8c$. In a relative spacetime the total speed of object O with respect to the frame of A is $\sqrt{(-0.6)^2c^2 + 0.8^2c^2} = c$. Object B is moving with respect to the frame of A with spatial speed c in the x direction. Its proper time speed is 0. The total speed of object B with respect to A therefore is $\sqrt{c^2 + 0} = c$.

If spacetime is relative, then it follows from figure 1.7 that the proper time of A lags with respect to the proper time of O, while it follows from figure 1.8 that the proper time of O lags with respect to the proper time of A. That is, if the present Euclidean spacetime is relative, then we are facing the twin paradox just as in SRT. The twin paradox does not occur in a Euclidean spacetime with a preferred frame.

We also consider the situation where B is an observer, see figure 1.9. The frame of observer B is denoted as $c\tau_B$ for the time axis and x_B for the x-axis. In a relative spacetime the object O is moving with respect to the frame of B with spatial speed $-0.8c$ in the x direction. Since its proper time is negative in the frame of B its proper time speed is $-0.6c$. The total speed of object O with respect to B is $\sqrt{(-0.8)^2c^2 + (-0.6)^2c^2} = c$.

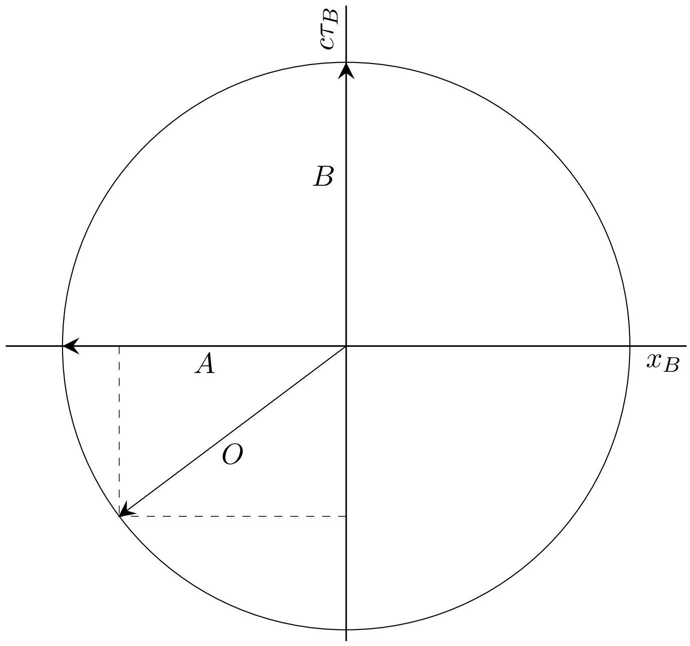

Figure 1.9: Two different objects, O and A as observed by observer B in a relative Euclidean spacetime.

Object A is moving with respect to the frame of B with spatial speed $-c$ in the x direction. Its proper time speed is 0. The total speed of object A with respect to B is $\sqrt{c^2 + 0} = c$.

If the Euclidean frames are relative, then observer O will regard object A as a particle and object B as an antiparticle, while observer A will regard object B as a photon and object O as a particle moving in the negative x-direction, and observer B will regard object O as an antiparticle moving in the negative x-direction and object A as a photon moving in the negative x direction. That is, in a relative Euclidean spacetime the character of an object depends on the frame of the observer. Such a character paradox does not occur if there is a preferred frame.

To illustrated another consequence we consider a collision of two particles, P and Q, with respect to two different frames, the frame of O and the frame of A. To show the situation the frames of both O and A are drawn in one diagram, see figure 1.10.

With respect to the frame of observer O the collision occurs when P and Q have the same x-coordinate. With respect to the frame of observer A the collision occurs when P and Q have different x-coordinates in the frame of A. Such a distant collision does not occur if there is a preferred frame of reference.

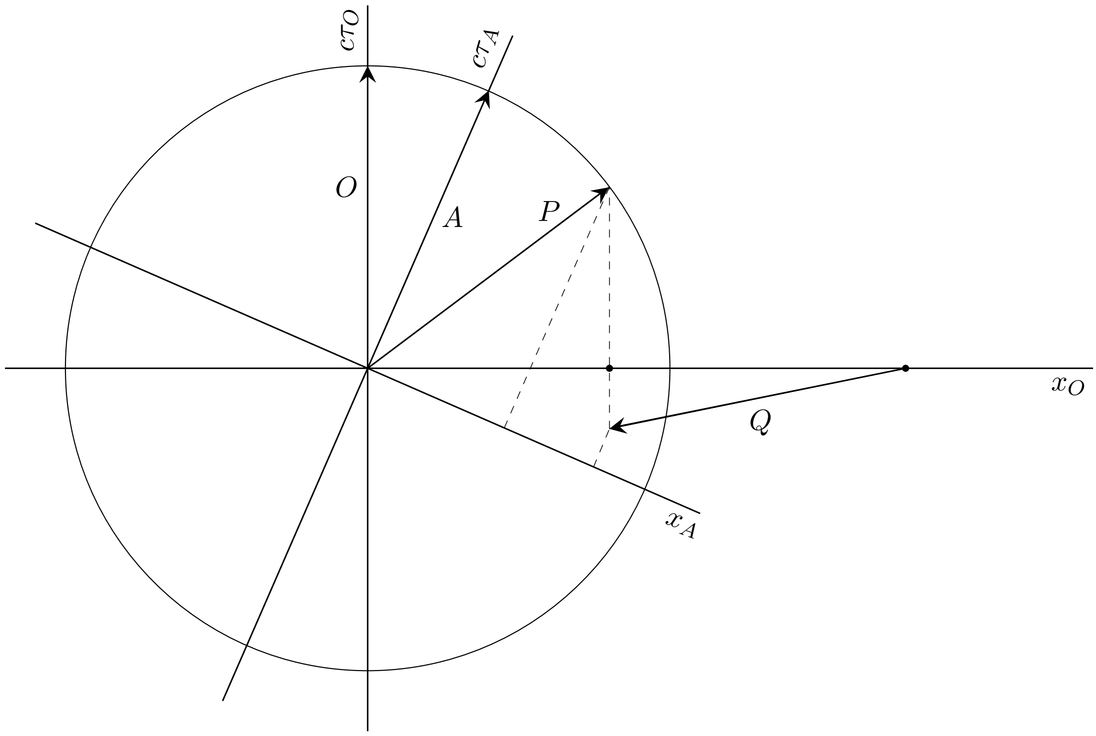

Figure 1.10: Two different observers, O and A, observing the collision of particle P with particle Q in a relative Euclidean spacetime.

1.5 Absolute Euclidean spacetime

From now on we will assume that the Euclidean spacetime has a preferred frame. Such a spacetime will be denoted as an absolute Euclidean spacetime (AEST), the preferred frame will also be denoted as the absolute rest frame and the time of a clock at rest in the preferred frame will also be denoted as the absolute time. Please note that the word 'absolute' only refers to the existence of a preferred frame of reference. In the present context it certainly does not refer to a classic situation where all clocks run equally fast. If we speak about 'absolute time' we just mean the time as indicated by a clock at rest in the preferred frame.

In the AEST are x, y and z, the space coordinates and is $c\tau$ the fourth coordinate. For the present purpose we will denote the system of the observer as a subscript and the observed object as a superscript. Since the fourth coordinate is independent of the observer, it is invariant, a subscript is redundant. Therefore the subscript will be left for the fourth coordinate. For instance, the coordinates of an object A with respect to absolute observer O are written as x_O^A, y_O^A, z_O^A and $c\tau^A$. Similarly, the coordinates of an object B with respect to observer O are written as x_O^B, y_O^B, z_O^B and $c\tau^B$. An absolute observer is at rest with respect to the preferred frame. Suppose A is an observer moving with constant speed $v = x_O^A/\tau^O$ with respect to the preferred frame. The coordinates of B with respect to A are denoted as x_A^B, y_A^B, z_A^B and $c\tau^B$. For the derivation of the relation between the coordinates of B with respect to A and the coordinates of B with respect to O and the coordinates of A with respect to O we restrict to one spatial direction, the x-direction, for convenience. For the x-coordinate of B with respect to A the Galilean relation would be $x_A^B = x_O^B - x_O^A$. The experiment of Michelson and Morley can only be explained in a space with a preferred frame if one assumes a physical length contraction by a factor

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} . \tag{1.20}$$

If the yardstick of observer A has shrunk with a factor γ, the measured distances are a factor γ larger. Adjusted for the length contraction the expression for x_A^B is

$$x_A^B = \gamma \left(x_O^B - x_O^A \right) . \tag{1.21}$$

By means of $v = x_O^A/\tau^O$ it can also be written as

$$x_A^B = \gamma \left(x_O^B - \frac{v}{c} c\tau^O \right) . \tag{1.22}$$

The latter is x_A^B as a function of x_O^B, τ^O and v. We recognise it as one part of the Lorentz transformation [9] . To derive the other part of the Lorentz transformation we have to consider clock synchronisation.

1.6 Clock synchronisation

To analyse the consequences of clock synchronisation we consider the following measurement of the speed of light in one direction, say the x-direction. Suppose a photon moves in the x-direction from one end of a moving rod to the other end where a mirror reflects the photon. From there the photon moves back to the beginning of the rod. Let the restframe of the rod be A. Again we let the velocity of A be constant. Due to the length contraction the length of the rod is l/γ, where l is the length of the rod if it would be at rest in the absolute rest frame. The absolute time lapse a photon needs to travel from one end of the rod to the other end is $\Delta_1 = \dfrac{l}{\gamma(c-v)}$. For the way back this is $\Delta_2 = \dfrac{l}{\gamma(c+v)}$. The total time interval Δ_{12} a photon needs to travel to and fro is

$$\Delta_{12} = \Delta_1 + \Delta_2 = \frac{l}{\gamma(c-v)} + \frac{l}{\gamma(c+v)} = \frac{2\,c\,l}{\gamma(c^2-v^2)} = \frac{2\,l\,\gamma}{c} . \tag{1.23}$$

The latter interval of time is according to the clock of absolute rest frame O. Since a clock in A runs slower by a factor γ the total to and fro time interval according to observer A is simply given by $\Delta_{12}/\gamma = 2\,l/c$. Since the yardstick of the observer A has shrunk by the same amount γ as the rod, the length of the rod with respect to A is $\gamma\,l/\gamma = l$. As a consequence, the speed of light as observed by A in the to and fro experiment becomes $\dfrac{2\,l}{\Delta_{12}/\gamma} = \dfrac{2\,l}{2\,l/c} = c$. That is, an observer at rest with respect to the rod measures with the to and fro experiment a speed of light equal to c. The present analysis makes clear that the invariance of the speed of light is not a property of spacetime. Instead , it is a consequence of the way we conduct light speed measurements.

Suppose we desire the speed of light also to be c if the photon travels only one way from one end of the rod to the other end. In accordance with the Einsteinian clock synchronisation we take two clocks, one at the mirror and one at the other end of the rod, and synchronise the clock at the mirror by an amount of ϵ such that the one way time intervals are half the time interval a photon needs to travel to and fro. Thus

$$\Delta_1 - \epsilon = \frac{l}{\gamma(c-v)} - \epsilon = \frac{\gamma\, l}{c} \tag{1.24}$$

and

$$\Delta_2 + \epsilon = \frac{l}{\gamma(c+v)} + \epsilon = \frac{\gamma\, l}{c}\,. \tag{1.25}$$

Solving for ϵ we obtain

$$\epsilon = \frac{\gamma\, l\, v}{c^2}\,. \tag{1.26}$$

Without the clock synchronisation we would we would have $\tau^A = \frac{\tau^O}{\gamma} = \gamma\left(\tau^O - \frac{v\,x^A_O}{c^2}\right)$. However, with the clock synchronisation the observed time interval becomes

$$\tau^A = \frac{\tau^O - \epsilon}{\gamma} = \gamma\left(\tau^O - \frac{v\,x^A_O}{c^2} - \frac{l\,v}{\gamma c^2}\right)\,. \tag{1.27}$$

Now we let observer A observe an object B. For the instant at which the spatial distance between A and B equals the length l/γ of the shrunken rod, we can substitute $l = \gamma(x^B_O - x^A_O)$. The result is

$$\tau^A = \gamma\left(\tau^O - \frac{v\,x^B_O}{c^2}\right)\,. \tag{1.28}$$

We recognise it as the other part of the Lorentz transformation.

1.7 Lorentz transformation

Due to the length contraction of rods and yardsticks, due to the time dilation of clocks moving with respect to the absolute restframe and due to clock synchronisation we arrived at the Lorentz transformations, see the equations (1.22) and (1.28). To simplify the notation we will leave the subscript if the observer is the absolute observer. Thus x^A means x^A_O and x^B means x^B_O. In addition, we will write the proper time of the absolute observer as t and the proper time of A as t^A, thus t means τ^O and t^A means τ^A. With this notation the transformations (1.22) and (1.28) read

$$x_A^B = \gamma\left(x^B - \frac{v}{c} c\, t\right) , \tag{1.29}$$

$$c\, t^A = \gamma\left(c\, t - \frac{v}{c} x^B\right) . \tag{1.30}$$

Infinitesimally, this is

$$\mathrm{d}x_A^B = \gamma(v)\left(\mathrm{d}x^B - \frac{v}{c} c\, \mathrm{d}t\right) , \tag{1.31}$$

$$c\, \mathrm{d}t^A = \gamma(v)\left(c\mathrm{d}t - \frac{v}{c}\, \mathrm{d}x^B\right) . \tag{1.32}$$

We have written the relativistic factor $(1 - v^2/c^2)^{-1/2}$ as $\gamma(v)$ in order to avoid confusion with other relativistic factors which will soon appear. The division of equation (1.31) by equation (1.32) leads to

$$v_A^B = \frac{w - v}{1 - wv/c^2} , \tag{1.33}$$

where $v_A^B = \frac{\mathrm{d}x_A^B}{\mathrm{d}t^A}$ is the velocity of B as observed by A, where $w = \mathrm{d}x^B/\mathrm{d}t$ is the velocity of B with respect to the absolute rest frame, and where v is the velocity of A with respect to the absolute rest frame. Similarly, for the velocity $v_B^F = \frac{\mathrm{d}x_F^B}{\mathrm{d}t^F}$ of an object F with respect to B, we have

$$v_B^F = \frac{u - w}{1 - uw/c^2} , \tag{1.34}$$

where w is the velocity of B with respect to the absolute rest frame, and where u is the velocity of F with respect to the absolute rest frame. The latter two equations can be elaborated to

$$v_A^F = \frac{v_A^B + v_B^F}{1 + v_A^B v_B^F/c^2} , \tag{1.35}$$

in which we recognise Einstein's addition theorem for velocities.

The equations (1.31) and (1.32) are transformations from frame O to frame A. For the same object B the transformation from frame O to a frame F then is

$$\mathrm{d}x_F^B = \gamma(u)\left(\mathrm{d}x^B - \frac{u}{c} c\, \mathrm{d}t\right) , \tag{1.36}$$

$$c\, \mathrm{d}t^F = \gamma(u)\left(c\mathrm{d}t - \frac{u}{c}\, \mathrm{d}x^B\right) , \tag{1.37}$$

where $\gamma(u) = (1 - u^2/c^2)^{-1/2}$ with u the velocity of F with respect to the absolute frame.

For the present AEST approach the question arises whether the Lorentz transformation also holds from a non absolute frame, say F to, for instance, frame A. For a positive answer there should hold

$$\mathrm{d}x_A^B = \gamma(v_F^A)\left(\mathrm{d}x_F^B - \frac{v_F^A}{c} c\, \mathrm{d}t^F\right) , \tag{1.38}$$

$$c\,\mathrm{d}t^A = \gamma(v_F^A)\left(c\mathrm{d}t^F - \frac{v_F^A}{c}\,\mathrm{d}x_F^B\right)\,, \tag{1.39}$$

where $\gamma(v_F^A) = (1-(v_F^A)^2/c^2)^{-1/2}$ with v_F^A the velocity of A with respect to the the frame of F. Substitution of the equations (1.36) and (1.37) into the equations (1.38) and (1.39) indeed leads to the equations (1.31) and (1.32). That is, even in an AEST we can arrive at the Lorentz transformation with respect to any observer. However, in an AEST the Lorentz transformation and the addition of velocities are not properties of spacetime. It are the consequences of an artificial clock synchronisation. Moreover, from the AEST point of view the Lorentz transformation is not a full coordinate transformation. Instead it is a transformation between the spatial coordinates and the parameter time.

1.8 Diagrams

In a Minkowski diagram we only see the spatial coordinates and not the proper times. The additional axis shows only the proper time of the absolute rest frame. In the AEST diagram we see all four coordinates, while the parameter time is absent. In the AEST the proper time of the absolute rest frame is used as a parameter time t for the time order of events and the determination of simultaneity. In order to get the full picture one can draw the t axis in addition to the spatial and proper time axes. For three spatial axes it would lead to a 5 dimensional diagram which is hard to visualise. If we confine to a single spatial axis, say x, then the full drawing is three dimensional and can be shown on a plane sheet of paper. For the hypothetical situation where different objects move off simultaneously from the origin with constant velocity, the graphical representation becomes as in figure 1.11.

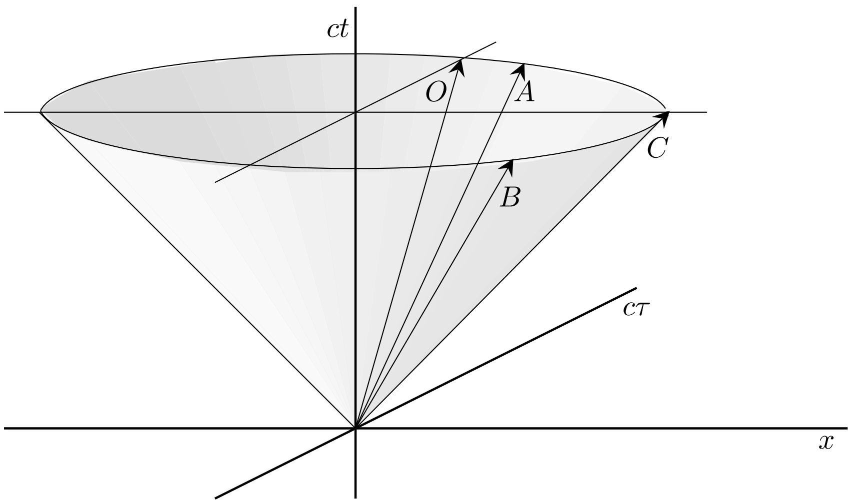

Figure 1.11: The world lines of various inertial objects O, A, B and C in an $(x, c\tau, ct)$ diagram.

Although the diagram in figure 1.11 is a $(x, c\tau, ct)$ diagram, the ct axis is not a spacetime axis. It is just a parameter axis for the time ordering of the coordinates of the paths: $x(t)$ and $c\tau(t)$. Since O, A, B and C move simultaneously through the origin, their coordinates are at each instant of time on a circle with radius ct. As a consequence, the simultaneous positions of O, A, B and C lie on a cone.

Taking a top view of the $(x, c\tau, ct)$ diagram we obtain the AEST diagram, see figure 1.12.

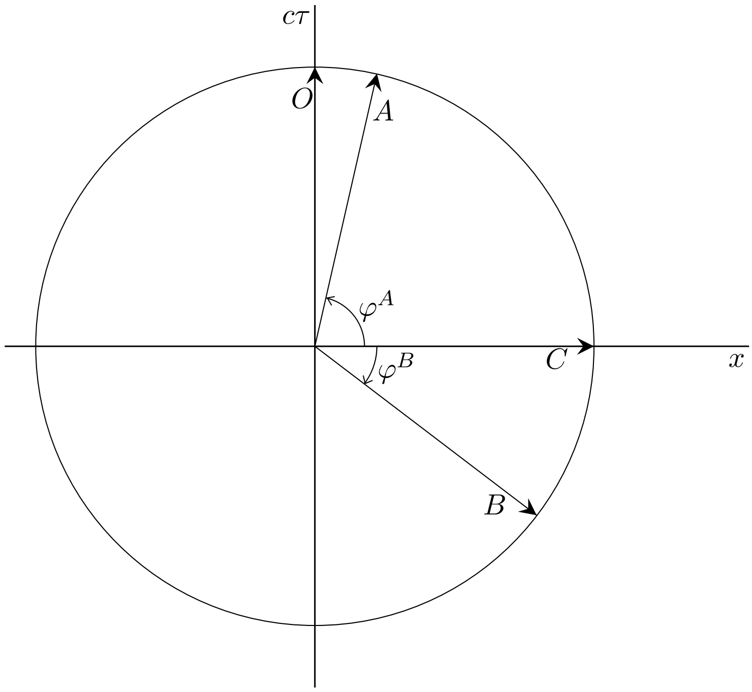

Figure 1.12: AEST diagram for four inertial objects O, A, B and C.

The endpoints of the paths of O, A, B and C lie on a circle. Circles in $(x, c\tau)$ diagrams has also been applied by Epstein [10]. In figure 1.12 the object O is at rest with respect to the absolute restframe. Its spatial velocity is $v = 0$. Its proper time velocity is $w = c\,\mathrm{d}\tau^O/\mathrm{d}t = c$. For object A we have $x^A(t) = c\,t\cos\varphi^A$ and $c\tau^A(t) = c\,t\sin\varphi^A$. Its spatial velocity and proper time velocity are $v = c\cos\varphi^A$ and $w = c\sin\varphi^A$ respectively. Object C is a photon moving in the positive x-direction. Its spatial velocity and proper time velocity are $v = c$ and $w = 0$ respectively. For object B we have $x^B(t) = c\,t\cos\varphi^B$ and $c\tau^B(t) = c\,t\sin\varphi^B$. Its spatial velocity and proper time velocity are $v = c\cos\varphi^B$ and $w = c\sin\varphi^B$ respectively. For object B we have $\varphi^B < 0$. That is, its proper time is negative and therefore it is regarded as an antiparticle.

The proper time velocity of A can be written as a function of the spatial velocity of A. That is, $\mathrm{d}\tau/\mathrm{d}t = \sin\varphi^A = \left(1 - \cos^2\varphi^A\right)^{1/2}$. Inserting $\sin\varphi^A = v/c$ yields $\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - v^2/c^2} = \mathrm{d}t/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. For an antiparticle, such as B, one should take the negative square root: $d\tau = -dt\sqrt{1 - v^2/c^2}$. The present analysis reveals the trigonometric nature of the relativistic factor.

Taking a frontview of the $(x, c\tau, ct)$ diagram we obtain the Minkowski diagram, see figure 1.13

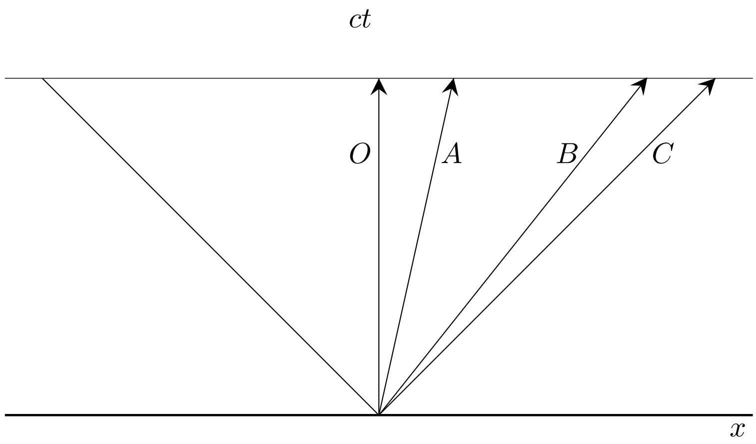

Figure 1.13: Minkowski diagram for four inertial objects O, A, B and C.

The front view illuminates why there is a light cone in the Minkowski diagram and a gap outside the light cone. From the paths in the Minkowski diagrams one can not tell whether an object is a particle or an antiparticle. Alternatively, the proper time of objects are absent in a Minkowski diagram. The analyses also makes clear that the Minkowski diagram is not a spacetime diagram. It is a space diagram extended with a parameter time axis. In the SRT the parameter time is taken as the fourth coordinate. In the SRT the role of the parameter time is ambiguous. On one hand it is a parameter for the time ordering, on the other hand it is a fourth coordinate. The consequence is that in the SRT the fourth coordinates of simultaneous events all have the same value. Since in the SRT the proper times of the objects acts as parameter times, there are as many parameters as there are objects. In the AEST there is a single parameter time to track all the moving objects, while each object has its own individual fourth coordinate. Simultaneous fourth coordinates can differ from object to object or from event to event. Briefly speaking, in the AEST there is a single parameter time for (the time ordering of) all the events, while there are as many fourth coordinates as there are events.

Chapter 2

Light speed experiments

2.1 Introduction

The present AEST is based on a physical length contraction and a physical time dilation. Together with an artificial clock synchronisation, one obtains in an AEST the same addition rule of relative velocities and the same Lorentz equations. In the AEST the addition rule for relative velocities is a consequence of an artificial clock synchronisation. This means that the addition rule for velocities is not of application to light speed experiments based on the shift of interference fringes due to a shift of the relative phase between two laser beams which follow different paths. An example of this is the Michelson and Morley experiment which can be explained in an AEST by means of a physical length contraction. A third example is Fizeau's experiment for the velocity of light in a moving medium. The result of the Fizeau experiment was already predicted by Fresnel. It was found that the difference between the velocity of light in a medium at rest and the velocity of light in a moving medium is only a fraction $\left(1 - \frac{1}{n^2}\right)$ of the velocity of the moving medium. The fraction is known as the Fresnel drag coefficient. The n is the index of refraction. Zeeman has observed that Fizeau's result should be slightly modified with a dispersion term. We will give an AEST explanation for the results of the Fizeau experiment, including the dispersion term.

2.2 Michelson and Morley experiment

In the Michelson and Morley experiment [11] a light beam is split in two perpendicular directions by means of a beam splitter. After been reflected by a mirror the two beams are recombined. The interference of the recombined beams leads to a fringe pattern. The setup is shown in figure 2.1. If the apparatus is at rest with respect to the absolute restframe, both arms have equal length L. Then both paths have length $2L$, from beam splitter to mirror and back to beamsplitter.

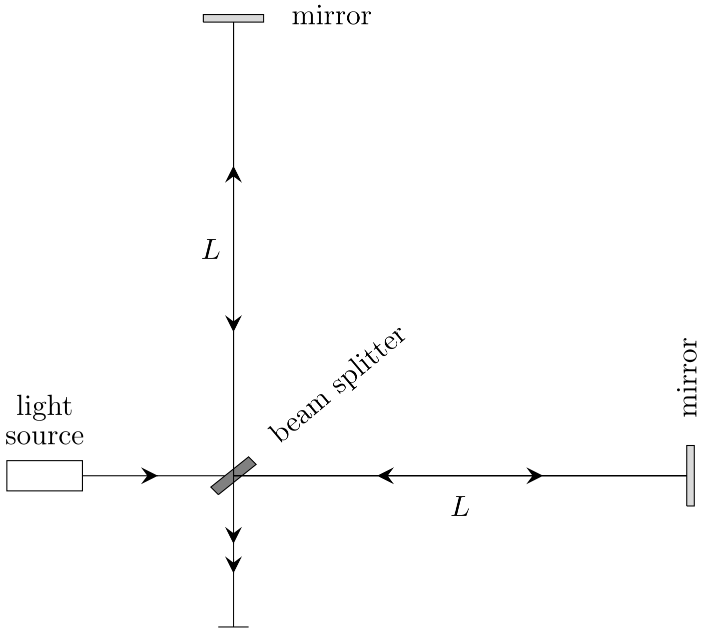

Figure 2.1: Setup of Michelson and Morley experiment. In rest with respect to the preferred frame both arms have length L.

If the apparatus is moving with a velocity $\vec{v}$ with respect to the absolute restframe, the lengths of the arms will be smaller than L due to length contraction. Next to this, the mirrors shift while the light is moving from the beam splitter to the mirrors. On the way back the beam splitter has moved while the light is moving from the mirrors to the beam splitter. The situation is shown in figure 2.2.

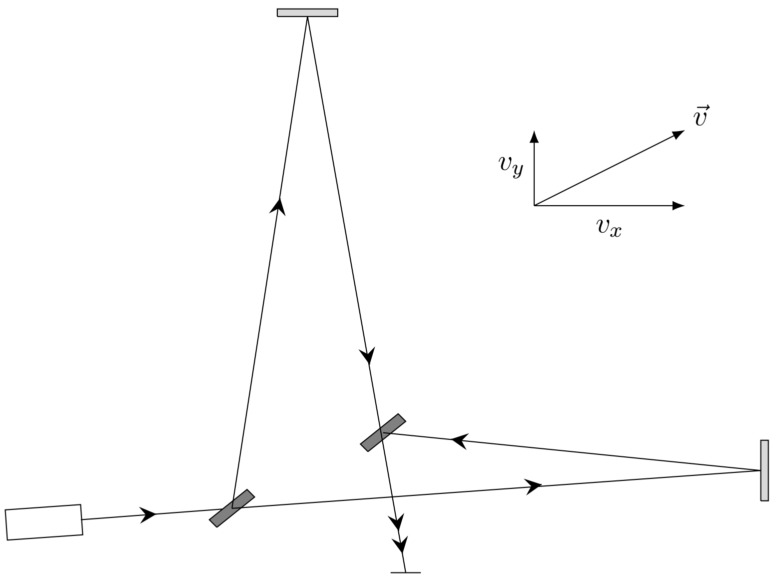

Figure 2.2: Michelson and Morley experiment moving with respect to the preferred frame.

Because of the velocity component v_y in the y-direction, the length of the vertical arm has shrunk by a factor $1/\gamma(v_y) = \sqrt{1 - v_y^2/c^2}$.
If t_{up} is the time a photon takes to move from the beamsplitter to the mirror at the vertical arm, we have

$$\left(L\sqrt{1 - \frac{v_y^2}{c^2}} + v_y t_{up}\right)^2 + v_x^2 t_{up}^2 = c^2 t_{up}^2 \,, \tag{2.1}$$

where $v_y t_{up}$ and $v_x t_{up}$ are the shifts in the y and x-direction because of the translation of the apparatus during t_{up}. It can be elaborated to

$$\left(c^2 - v^2\right) t_{up}^2 - 2v_y L\sqrt{1 - \frac{v_y^2}{c^2}}\, t_{up} - L^2\left(1 - \frac{v_y^2}{c^2}\right) = 0\,, \tag{2.2}$$

where $v^2 = v_x^2 + v_y^2$. It is a quadratic equation for t_{up} with solution

$$c\,t_{up} = \frac{L\left(v_y + \sqrt{c^2 - v_x^2}\right)\sqrt{c^2 - v_y^2}}{c^2 - v^2}\,. \tag{2.3}$$

If t_{down} is the time a photon takes to move from the mirror at the vertical arm back to the beam splitter, we have

$$\left(L\sqrt{1 - \frac{v_y^2}{c^2}} - v_y t_{down}\right)^2 + v_x^2 t_{down}^2 = c^2 t_{down}^2 \,, \tag{2.4}$$

where $v_y t_{down}$ and $v_x t_{down}$ are the shifts in the y and x-direction because of the translation of the apparatus during t_{down}. It can be elaborated to

$$\left(c^2 - v^2\right) t_{down}^2 + 2v_y L\sqrt{1 - \frac{v_y^2}{c^2}}\, t_{down} - L^2\left(1 - \frac{v_y^2}{c^2}\right) = 0\,. \tag{2.5}$$

The quadratic equation for t_{down} has the solution

$$c\,t_{down} = \frac{L\left(-v_y + \sqrt{c^2 - v_x^2}\right)\sqrt{c^2 - v_y^2}}{c^2 - v^2}\,. \tag{2.6}$$

For the total time back and forth we obtain

$$t_1 = t_{up} + t_{down} = \frac{2L}{c}\frac{\sqrt{c^2 - v_x^2}\sqrt{c^2 - v_y^2}}{c^2 - v^2} = \frac{2L}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)}\,. \tag{2.7}$$

The coordinates of the photon at that time are

$$(x_1, y_1) = (v_x t_1, v_y t_1) = \left(2L\frac{v_x}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)}, 2L\frac{v_y}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)}\right)\,. \tag{2.8}$$

If t_{right} is the time a photon takes to move from the beamsplitter to the mirror at the horizontal arm, we have

$$\left(L\sqrt{1-\frac{v_x^2}{c^2}}+v_x t_{right}\right)^2+v_y^2 t_{right}^2=c^2 t_{right}^2\,, \tag{2.9}$$

where $v_y t_{right}$ and $v_x t_{right}$ are the shifts in the y and x-direction because of the translation of the apparatus during t_{right}. It can be elaborated to

$$\left(c^2-v^2\right) t_{right}^2-2v_x L\sqrt{1-\frac{v_x^2}{c^2}}\,t_{right}-L^2\left(1-\frac{v_x^2}{c^2}\right)=0\,. \tag{2.10}$$

Also here $v^2=v_x^2+v_y^2$. The quadratic equation for t_{right} has the solution

$$c\,t_{right}=\frac{L\left(v_x+\sqrt{c^2-v_y^2}\right)\sqrt{c^2-v_x^2}}{c^2-v^2}\,. \tag{2.11}$$

If t_{left} is the time a photon takes to move from the mirror at the horizontal arm back to the beam splitter, we have

$$\left(L\sqrt{1-\frac{v_x^2}{c^2}}-v_x t_{left}\right)^2+v_y^2 t_{left}^2=c^2 t_{left}^2\,, \tag{2.12}$$

where $v_y t_{left}$ and $v_x t_{left}$ are the shifts in the y and x-direction because of the translation of the apparatus during t_{left}. It can be elaborated to

$$\left(c^2-v^2\right) t_{left}^2+2v_x L\sqrt{1-\frac{v_x^2}{c^2}}\,t_{left}-L^2\left(1-\frac{v_x^2}{c^2}\right)=0\,. \tag{2.13}$$

The quadratic equation for t_{left} has the solution

$$c\,t_{left}=\frac{L\left(-v_x+\sqrt{c^2-v_y^2}\right)\sqrt{c^2-v_x^2}}{c^2-v^2}\,. \tag{2.14}$$

For the total time back and forth we obtain

$$t_2=t_{right}+t_{left}=\frac{2L}{c}\frac{\sqrt{c^2-v_x^2}\sqrt{c^2-v_y^2}}{c^2-v^2}=\frac{2L}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)}\,. \tag{2.15}$$

This is identical to the time t_1 for the vertical arm. The coordinates of the photon at that time is

$$(x_2,y_2)=(v_x t_2,v_y t_2)=\left(2L\frac{v_x}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)},2L\frac{v_y}{c}\frac{\gamma^2(v)}{\gamma(v_x)\gamma(v_y)}\right)\,. \tag{2.16}$$

The coordinates are identical to the ones for the vertical arm. Most important is the absence of a difference in the total times of the two arms: $t_2-t_1=0$. Hence, the fringes pattern will not shift when the direction or magnitude of the velocity of the apparatus changes. That is, the null result of the Michelson and Morley experiment can be explained in an AEST by means of a physical length contraction.

2.3 Kennedy-Thorndike experiment

In the Michelson and Morley experiment both arms have equal length L. The Kennedy-Thorndike experiment [12] differs from the Michelson and Morley experiment in that arms of different length is used. Let L_1 and L_2 be the length of the vertical and horizontal arm respectively, when they are at rest with respect top the preferred frame. The analysis is almost the same as for the Michelson and Morley experiment. Substituting L_1 for L in equation (2.7) gives for the back and forth time of the vertical arm

$$t_1 = \frac{2L_1}{c} \frac{\sqrt{c^2 - v_x^2}\sqrt{c^2 - v_y^2}}{c^2 - v^2} . \tag{2.17}$$

Substituting L_2 for L in equation (2.15) gives for the back and forth time of the horizontal arm

$$t_2 = \frac{2L_2}{c} \frac{\sqrt{c^2 - v_x^2}\sqrt{c^2 - v_y^2}}{c^2 - v^2} . \tag{2.18}$$

We see for arms of different length there is a difference in the total times:

$$t_2 - t_1 = 2\frac{L_2 - L_1}{c} \frac{\sqrt{c^2 - v_x^2}\sqrt{c^2 - v_y^2}}{c^2 - v^2} . \tag{2.19}$$

The variation of the total time difference depends on the velocity $\vec{v}$. For the fringe shift we have to consider the variation of the total time difference.
If $v_x = 0$ or $v_y = 0$ then

$$t_2 - t_1 = 2c\frac{L_2 - L_1}{c^2 - v^2}\sqrt{1 - \frac{v^2}{c^2}} \approx 2c\frac{L_2 - L_1}{c^2 - v^2}\left(1 - \frac{v^2}{2c^2} - \frac{v^4}{8c^4} - \frac{v^6}{16c^6} + ...\right) . \tag{2.20}$$

If $v_x = v_y = v/\sqrt{2}$ then

$$t_2 - t_1 = 2c\frac{L_2 - L_1}{c^2 - v^2}\left(1 - \frac{v^2}{2c^2}\right) . \tag{2.21}$$

Subtracting the latter two equations from each other, we obtain for the maximum variation

$$c\delta t \approx \frac{v^4}{4c^4}\left(L_2 - L_1\right) . \tag{2.22}$$

The velocity of the earth orbiting around the sun is of order $\frac{v}{c} \approx 10^{-4}$. For a typical arm length difference of $L_2 - L_1 = 0.2$ m the maximum variation is $c\delta t = 5 \cdot 10^{-18}$ m. For a typical wavelength of $5 \cdot 10^{-7}$ m the maximum variation is 10^{-11} wavelength, which is practically zero. That is, the practical null result of the Kennedy-Thorndike experiment can be explained in an AEST by means of a physical length contraction.

2.4 The Fizeau experiment in relativity theory

In a vacuum the velocity of light is c. Let the velocity of light in a medium at rest be c'. There holds: $c' = c/n(0)$, where $n(0)$ is the index of refraction of a medium at rest with respect to the observer. Next we consider a medium moving with velocity v with respect to the same observer. We assume the photon will move in the same direction as the medium. Let the velocity of light in the moving medium be c''. There holds $c'' = c/n(v)$, where $n(v)$ is the index of refraction of the moving medium with respect to the observer. In the SRT the addition theorem for velocities delivers the following relation:

$$c'' = \frac{v + c^*}{1 + vc^*/c^2}\,, \tag{2.23}$$

where v is the velocity of the medium and c^* is the velocity of light in the moving medium with respect to an observer moving with the medium.

2.4.1 Nondispersive medium

If the index of refraction does not depend on the frequency of light, the principle of relativity implies that c^* can be taken equal to c', thus $c^* = c/n(0)$. Substituting of $c'' = c/n(v)$ and $c^* = c/n(0)$ in equation (2.23) and neglecting terms of order v^2/c^2 we obtain

$$\frac{1}{n(v)} \approx \frac{1}{n(0)} + \frac{v}{c}\left(1 - \frac{1}{n(0)^2}\right)\,. \tag{2.24}$$

The latter equals the experimental result as it was found by Fizeau [13, 14] .

2.4.2 Dispersive medium

If the index of refraction does depend on the frequency of light, the frequency f^* of light with respect to an observer moving with the moving medium differs from the frequency f of the light with respect to the observer at rest. There holds

$$f^* = f\left(1 - \frac{v}{c''(f)}\right)\gamma\,. \tag{2.25}$$

The factor $\left(1 - \frac{v}{c''(f)}\right)$ is the classical Doppler effect. The factor $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ occurs because of the dilation of time. To order v/c the equation (2.25) yields

$$f^* - f \approx -\frac{vf}{c''(f)}\,. \tag{2.26}$$

For the dispersive case the principle of relativity implies that $c^*(f^*)$ should be taken equal to $c'(f^*)$. In order to obtain $c'(f^*)$ from $c'(f)$ we consider the Taylor expansion

$$c'(f^*) = c'(f) + \frac{\mathrm{d}c'(f)}{\mathrm{d}f}(f^* - f) + \ldots\,. \tag{2.27}$$

The Taylor expansion can also be written as

$$c'(f^*) = c'(f) + \frac{\partial c'(f)}{\partial n(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f}(f^* - f) + \dots , \tag{2.28}$$

where we used the notation $n(v, f)$ for an index of refraction which depends on the velocity of the medium and on the frequency of light. Substituting equation (2.26) into equation (2.28), we obtain for $c^*(f^*) = c'(f^*)$ to order v/c:

$$c^*(f^*) \approx c'(f) - \frac{\partial c'(f)}{\partial n(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f} \frac{vf}{c''(f)} . \tag{2.29}$$

Together with $c'(f) = \dfrac{c}{n(0,f)}$ and thus $\dfrac{\partial c'(f)}{\partial n(0,f)} = \dfrac{-c}{n^2(0,f)}$, the latter reads

$$c^*(f^*) \approx \frac{c}{n(0,f)} + \frac{c}{n^2(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f} \frac{vf}{c''(f)} . \tag{2.30}$$

The velocity of light in a moving medium with respect to an observer at rest is given by

$$c''(f) = \frac{c}{n(v,f)} , \tag{2.31}$$

with v the velocity of the moving medium. Substituting equation (2.30) and equation (2.31) into the addition theorem for velocities,

$$c''(f) = \frac{v + c^*(f^*)}{1 + v\, c^*(f^*)/c^2} , \tag{2.32}$$

and leaving the frequency argument in the index of refraction, we obtain to order v/c

$$\frac{1}{n(v)} = \frac{1}{n(0)} + \frac{v}{c}\left(1 - \frac{1}{n^2(0)} + \frac{f}{n(0)} \frac{\mathrm{d}n(0)}{\mathrm{d}f}\right) . \tag{2.33}$$

The latter equals the experimental result as it was found by Zeeman [15,16] .

2.5 The Fizeau experiment in an AEST

For the situation in an AEST, c is the velocity of light in vacuum, $c' = c/n(0)$ is the velocity of light in a medium at rest with respect to the preferred frame and $c'' = c/n(v)$ is the velocity of light in a medium moving with speed v with respect to the preferred frame. Also here we let the photon move in the same direction as the medium. Here $n(0)$ is the index of refraction of a medium at rest with respect to the preferred frame. Similarly, $n(v)$ is the index of refraction of the moving medium with respect to the preferred frame. In the AEST we can not apply the velocity addition theorem to experiments based on the shift of interference fringes. Instead, we will explore proper time. For a photon in a medium at rest there holds according to the AEST

$$\tau'^2 = t'^2/\sqrt{1 - c'^2/c^2} = t'^2/\sqrt{1 - 1/n^2(0)} . \tag{2.34}$$

For a photon in a medium moving with velocity v with respect to the preferred frame this is

$$\tau''^2 = \mathrm{d}t''^2/\sqrt{1-c''^2/c^2} = t''^2/\sqrt{1-1/n^2(v)}\,. \tag{2.35}$$

That is, in a medium a photon has a component in the proper time dimension. It is caused by the interaction of the photon with the 'molecules' of the medium. Each molecule contributes to the proper time of the passing photon. For $v << c$ the contribution of a single molecule at rest is not expected to differ substantially from the contribution of a single molecule moving with velocity v. If the contribution of a single molecule to the increase of proper time of the passing photon does depend on the velocity v of the medium than it can be readily assumed that it will not depend on the direction of the velocity of the medium. With this in mind we consider the Fizeau experiment in an AEST [17] .

Let a medium at rest have length L. The time it takes for the photon to travel through the medium is

$$t' = \frac{Ln(0)}{c}\,. \tag{2.36}$$

For the increase of the proper time of the photon we have

$$\tau' = \frac{t'}{\gamma(c')} = t'\sqrt{1-\frac{c'^2}{c^2}} = t'\sqrt{1-\frac{1}{n^2(0)}}\,. \tag{2.37}$$

Hence,

$$\tau' = \frac{Ln(0)}{c}\sqrt{1-\frac{1}{n^2(0)}}\,. \tag{2.38}$$

Next we let the same medium move with velocity v. For the time t'' a photon needs to travel through the moving medium, we have

$$\frac{L}{\gamma(v)} + vt'' = \frac{c}{n(v)}t'' \quad \rightarrow \quad t'' = \frac{L}{\gamma(v)\left(\frac{c}{n(v)} - v\right)}\,. \tag{2.39}$$

The vt'' it the shift of the medium during t''. The $\gamma(v)$ is because of the length contraction of the medium. In the moving medium the increase of the proper time of the photon is

$$\tau'' = \frac{t''}{\gamma(c'')} = \frac{L}{\gamma(v)\left(\frac{c}{n(v)} - v\right)}\sqrt{1-\frac{1}{n^2(v)}}\,. \tag{2.40}$$

2.5.1 Nondispersive medium

In the moving medium the photon experiences as many molecules as in the medium at rest. For a nondispersive medium this means that

$$\frac{\tau''}{\tau'} = 1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)\,. \tag{2.41}$$

Since terms of order v^2/c^2 can be neglected the AEST rule is

$$\tau'' = \tau' \,. \tag{2.42}$$

Substitution of equation (2.40) and equation (2.38) into equation (2.42) leads to

$$\frac{L}{\frac{c}{n(v)} - v}\sqrt{1 - \frac{1}{n^2(v)}} = \frac{Ln(0)}{c}\sqrt{1 - \frac{1}{n^2(0)}} \,. \tag{2.43}$$

It is reduced to

$$\frac{1}{n(0)}\sqrt{1 - \frac{1}{n^2(v)}} = \left(\frac{1}{n(v)} - \frac{v}{c}\right)\sqrt{1 - \frac{1}{n^2(0)}} \,. \tag{2.44}$$

Taking the square on both sides, we obtain

$$\frac{1}{n^2(0)}\left(1 - \frac{1}{n^2(v)}\right) = \left(\frac{1}{n(v)} - \frac{v}{c}\right)^2\left(1 - \frac{1}{n^2(0)}\right) \,. \tag{2.45}$$

The latter can be elaborated to

$$\frac{1}{n^2(v)} - \frac{2v}{c}\left(1 - \frac{1}{n^2(0)}\right)\frac{1}{n(v)} - \frac{1}{n^2(0)} + \frac{v^2}{c^2}\left(1 - \frac{1}{n^2(0)}\right) = 0 \,. \tag{2.46}$$

Taking the positive root of this quadratic equation for $1/n(v)$ and neglecting terms of order v^2/c^2, we finally get the experimental result of Fizeau:

$$\frac{1}{n(v)} = \frac{1}{n(0)} + \frac{v}{c}\left(1 - \frac{1}{n^2(0)}\right) \,. \tag{2.47}$$

2.5.2 Dispersive medium

As before we let f be the frequency of the photon with respect to the preferred frame and f^* be the frequency of the photon with respect to the moving medium. For a dispersive medium the index of refraction depends on the frequency of light. As a consequence the proper time of a photon moving through the medium depends on the frequency of light. With respect to the moving medium the frequency of light f^* differs from the frequency f with respect to a medium at rest. In an AEST the relation between f^* and f is the same as in SRT. Thus

$$f^* = f\left(1 - \frac{v}{c''(f)}\right)\gamma \,, \tag{2.48}$$

where the factor $\left(1 - \frac{v}{c''(f)}\right)$ is the classical Doppler effect and the factor $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the dilation of time. To order v/c the equation (2.25) yields

$$f^* - f \approx -\frac{vf\,n(0,f)}{c} \,. \tag{2.49}$$

For a dispersive medium we can not simply compare $\tau''(f^*)$ with $\tau'(f)$. Instead, we should take

$$\tau''(f^*) = \tau'(f^*) \,. \tag{2.50}$$

In order to obtain $\tau'(f^*)$ from $\tau'(f)$ we consider the Taylor expansion

$$\tau'(f^*) = \tau'(f) + \frac{\mathrm{d}\tau'(f)}{\mathrm{d}f}(f^* - f) + \dots . \tag{2.51}$$

The Taylor expansion can also be written as

$$\tau'(f^*) = \tau'(f) + \frac{\partial \tau'(f)}{\partial n(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f}(f^* - f) + \dots . \tag{2.52}$$

Also here we use the notation $n(v, f)$ for an index of refraction which depends on the velocity of the medium and on the frequency of light. Substituting equation (2.49) into equation (2.52), we obtain for $\tau'(f^*)$ to order v/c:

$$\tau'(f^*) \approx \tau'(f) - \frac{\partial \tau'(f)}{\partial n(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f} \frac{vf\,n(0,f)}{c} . \tag{2.53}$$

Substitution of the latter in equation (2.50) gives

$$\tau''(f^*) = \tau'(f) - \frac{\partial \tau'(f)}{\partial n(0,f)} \frac{\mathrm{d}n(0,f)}{\mathrm{d}f} \frac{vf\,n(0,f)}{c} . \tag{2.54}$$

Substituting equation (2.40) for $\tau''(f^*)$ and equation (2.38) for $\tau'(t)$ into the latter, we obtain

$$\frac{L\sqrt{1-\frac{1}{n^2(v)}}}{\gamma(v)\left(\frac{c}{n(v)} - v\right)} = \frac{Ln(0)}{c}\sqrt{1-\frac{1}{n^2(0)}} - \frac{Ln(0)}{c\sqrt{1-\frac{1}{n^2(0)}}} \frac{\mathrm{d}n(0)}{\mathrm{d}f} \frac{vf}{c} , \tag{2.55}$$

where we have left the frequency argument for brevity. To order v/c the latter is

$$\frac{\sqrt{1-\frac{1}{n^2(v)}}}{\frac{1}{n(v)} - \frac{v}{c}} = (1-\epsilon)\, n(0)\sqrt{1-\frac{1}{n^2(0)}} , \tag{2.56}$$

where ϵ is defined as

$$\epsilon = \frac{f}{1-\frac{1}{n^2(0)}} \frac{\mathrm{d}n(0)}{\mathrm{d}f} \frac{v}{c} . \tag{2.57}$$

The equation (2.56) can be rearranged to

$$\frac{1}{n(0)}\sqrt{1-\frac{1}{n^2(v)}} = (1-\epsilon)\left(\frac{1}{n(v)} - \frac{v}{c}\right)\sqrt{1-\frac{1}{n^2(0)}} . \tag{2.58}$$

Taking the square at both sides of the equation, we obtain to order v/c

$$\left(1 - 2\epsilon + \frac{2\epsilon}{n^2(0)}\right)\frac{1}{n^2(v)} - \left(\frac{2v}{c} - \frac{2v}{cn^2(0)}\right)\frac{1}{n(v)} - \frac{1}{n^2(0)} = 0 . \tag{2.59}$$

Taking the positive root of this quadratic equation for $1/n(v)$, we obtain to order v/c

$$\frac{1}{n(v)} = \frac{1}{n(0)} + \frac{v}{c}\left(1 - \frac{1}{n^2(0)}\right) + \frac{\epsilon}{n(0)}\left(1 - \frac{1}{n^2(0)}\right) . \tag{2.60}$$

By putting back the equation (2.57) we finally arrive at the experimental result of Zeeman:

$$\frac{1}{n(v)} = \frac{1}{n(0)} + \frac{v}{c}\left(1 - \frac{1}{n^2(0)} + \frac{f}{n(0)}\frac{\mathrm{d}n(0)}{\mathrm{d}f}\right) . \tag{2.61}$$

Between the brackets we could have written $n(v)$ instead of $n(0)$ since the difference between $1/n(v)$ and $1/n(0)$ is of order v/c.

2.5.3 Moving observer

The equation (2.61) is derived with respect to the absolute rest frame. In the SRT any inertial observer may regard himself as an observer at rest. In an AEST we do not have such a principle of relativity. In an AEST some further analysis is needed to find the relation between two moving mediums. We let the two mediums move in the same direction with different speeds: v and w with $w > v$. According to equation (2.61) there holds

$$\frac{1}{n(v)} = \frac{1}{n(0)} + \eta\frac{v}{c} \tag{2.62}$$

and

$$\frac{1}{n(w)} = \frac{1}{n(0)} + \eta\frac{w}{c}\,, \tag{2.63}$$

where η is the part between brackets in equation (2.61). From the latter two equations it follows

$$\frac{1}{n(w)} = \frac{1}{n(v)} + \eta\frac{w-v}{c}\,, \tag{2.64}$$

which we rewrite for our purpose to

$$\frac{c}{n(w)} - v = \frac{c}{n(v)} - v + \eta(w-v)\,. \tag{2.65}$$

The velocity of the photon in the fastest medium is $c/n(w)$ with respect to the absolute observer. Now we let an observer move with the same speed v and the same direction as the slower medium. With respect to the moving observer the velocity of the photon in the faster medium is to order v/c given by

$$\frac{c}{n^*(w-v)} = \frac{c}{n(w)} - v\,. \tag{2.66}$$

Because the effects of time dilation and length contraction are of order v^2/c^2, the first order relation between the speeds is just Galilean. The notation n^* is used when the index of refraction is with respect to the moving observer. The velocity of the photon in the slower medium is $c/n(v)$ with respect to the absolute observer. With respect to the moving observer the velocity of the photon in the slower medium is to order v/c

$$\frac{c}{n^*(0)} = \frac{c}{n(v)} - v\,. \tag{2.67}$$

Substitution of the latter two equations in equation (2.65) gives

$$\frac{c}{n^*(w-v)} = \frac{c}{n^*(0)} + \eta(w-v)\,. \tag{2.68}$$

By putting back the expression for η we get

$$\frac{1}{n^*(u)} = \frac{1}{n^*(0)} + \frac{u}{c}\left(1 - \frac{1}{n^{*2}(0)} + \frac{f}{n^*(0)}\frac{\mathrm{d}n^*(0)}{\mathrm{d}f}\right)\,, \tag{2.69}$$

where u is the relative velocity of the faster moving medium with respect to the moving observer: $u = w - v$. The latter equation shows that the equation (2.61) is also valid with respect to a moving observer. In other words, to order v/c there is relativity present in an AEST.

Chapter 3

Kinematics in an AEST

3.1 Four vectors

In the previous chapters experiments were considered for which the AEST predictions are identical to the SRT predictions. Apparently there is a lot of relativity present in an AEST. Nevertheless there are important fundamental differences. Because of these differences we can not apply the rules of the SRT for transformations to moving observers. Instead we will mainly restrict to physics with respect to the absolute restframe. Since relativity is present in an AEST the physics with respect to an observer which moves with respect to the absolute restframe will be to a large extent identical to the physics with respect to the absolute restframe. Although different from SRT the AEST is four dimensional. It therefore is convenient to use the four vector notation.

In the AEST there are three spatial coordinates. In the AEST the proper time of an object is its fourth coordinate. The position of an object in an AEST is a four vector: $(x, y, z, c\tau)$ or shortly x_μ. Thus $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = c\tau$. The actual coordinates of an object are parameterised by the time t of a clock at rest with respect to the AEST. Thus $(x(t), y(t), z(t), c\tau(t))$ or $x_\mu(t)$. The velocity of an object in an AEST is a four vector: $(\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, \mathrm{d}z/\mathrm{d}t, c\mathrm{d}\tau/\mathrm{d}t)$ or shortly u_μ. Thus $u_1 = \dot{x}$, $u_2 = \dot{y}$, $x_3 = \dot{z}$ and $x_4 = c\dot{\tau}$, where the dot denotes the derivative with respect to t. Also the velocities are parameterised by the time t of a clock at rest with respect to the AEST. Thus $u_\mu(t)$. Since in an AEST the metric is Euclidean there is no difference between u^μ and u_μ: $u_\mu = \delta_{\mu\nu} u^\nu$, where δ is the Euclidean metric, $\delta = \mathrm{diag}(1, 1, 1, 1)$. In fact we will never use the superscript for vectors. All AEST vectors will be denoted with a subscript. Summation is understood over repeated indices. The basic principle of the AEST is that in a free space everything moves with a four dimensional Euclidean velocity equal to the speed of light in free space. In four vector notation the principle reads

$$u_\mu u_\mu = c^2 \,, \tag{3.1}$$

where u_1, u_2 and u_3 are the spatial velocities and where u_4 is the proper time velocity. If v is the total spatial velocity, $u_1^2+u_2^2+u_3^2=v^2$, then $\frac{u_4}{c}=\sqrt{1-\frac{v^2}{c^2}}$. That is, the relativistic time dilation is a direct consequence of the equation (3.1). With the dot notation the equation (3.1) reads

$$\dot{x}^2+\dot{y}^2+\dot{z}^2+c^2\dot{\tau}^2=c^2\,. \tag{3.2}$$

The latter can also be expressed as

$$\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2+c^2\mathrm{d}\tau^2=c^2\mathrm{d}t^2\,. \tag{3.3}$$

The left side is the infinitesimal Euclidean displacement of the object:

$$\mathrm{d}s_{obj}^2=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2+c^2\mathrm{d}\tau^2\,. \tag{3.4}$$

Since $\mathrm{d}x=\mathrm{d}y=\mathrm{d}z=0$ for an observer at rest in the preferred frame, a so called absolute observer, the right side of equation (3.3) is the infinitesimal displacement of an absolute observer:

$$\mathrm{d}s_{obs}^2=c^2\mathrm{d}t^2\,. \tag{3.5}$$

3.2 Snell's law and the action principle

For the action principle in an AEST we start considering the path of a photon in two different mediums. The photon starts at point A in medium 1. A little later it arrives at point B in medium 2, see figure 3.1.

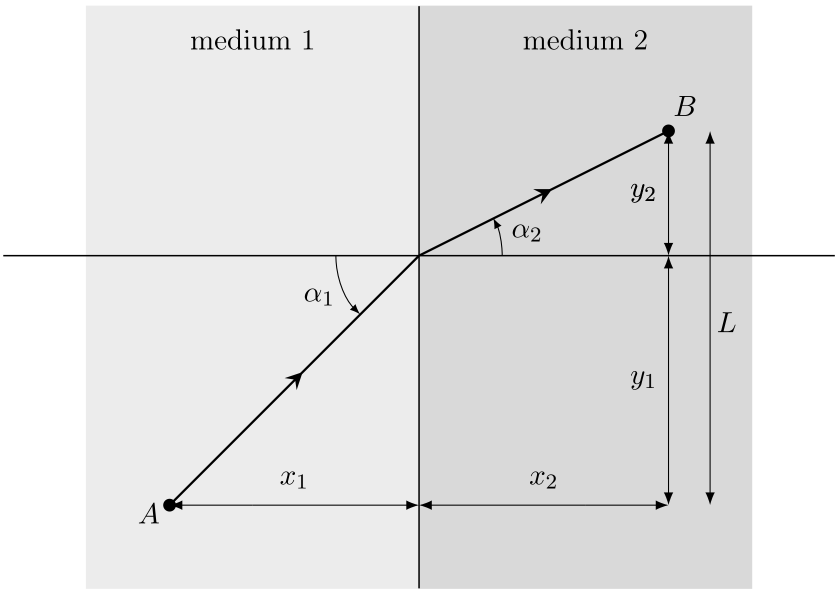

Figure 3.1: The path of a photon in two mediums

The distances the photon has traveled in medium 1 and medium 2 are

$$l_1 = \sqrt{x_1^2 + y_1^2} \tag{3.6}$$

and

$$l_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{x_2^2 + (L - y_1)^2} \tag{3.7}$$

respectively. The time t the photon takes to travel from A to B amounts to

$$t = \frac{l_1\, n_1}{c} + \frac{l_2\, n_2}{c} = \frac{n_1\sqrt{x_1^2 + y_1^2}}{c} + \frac{n_2\sqrt{x_2^2 + (L - y_1)^2}}{c}\,, \tag{3.8}$$

where n_1 and n_2 are the index of refraction of medium 1 and medium 2 respectively. Differentiation of the latter with respect to y_1 yields

$$\frac{\mathrm{d}t}{\mathrm{d}y_1} = \frac{n_1\, y_1}{l_1} - \frac{n_2\,(L - y_1)}{cl_2} = n_1 \sin\alpha_1 - n_2 \sin\alpha_2\,. \tag{3.9}$$

According to Fermat's principle the path of the photon is such that the interval of time is minimised. In other words, y_1 is such that $\frac{\mathrm{d}t}{\mathrm{d}y_1} = 0$. This implies that

$$\frac{n_1}{n_2} = \frac{\sin\alpha_2}{\sin\alpha_1}\,. \tag{3.10}$$

We clearly recognise Snell's law [18] .

In the theory of relativity particles are supposed to take the shortest possible path between two points with the length of the path being measured by the proper time of the particle. Alternatively, in the theory of relativity the action I of a particle is proportional to the integral of the infinitesimal Minkowski displacement ds. In the theory of relativity the infinitesimal distance ds is c times the proper time interval dτ of the particle. Thus

$$I \propto \int \mathrm{d}s = c \int \mathrm{d}\tau\,. \tag{3.11}$$

The path of the particle minimises the action. Now we show it does not make sense to apply equation (3.11) to the refraction of light. The proper time the photon takes to travel from A to B in figure 3.1 amounts to

$$\tau = \frac{l_1\, n_1}{c\gamma_1} + \frac{l_2\, n_2}{c\gamma_2}\,, \tag{3.12}$$

where $\gamma_1 = \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}}$ and $\gamma_2 = \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}$. Substitution of $v_1 = \frac{c}{n_1}$, $v_2 = \frac{c}{n_2}$, $l_1 = \sqrt{x_1^2 + y_1^2}$ and $l_2 = \sqrt{x_2^2 + (L - y_1)^2}$ leads to

$$\tau = \frac{\sqrt{(n_1^2 - 1)(x_1^2 + y_1^2)}}{c} + \frac{\sqrt{(n_2^2 - 1)(x_2^2 + (L - y_1)^2)}}{c}\,. \tag{3.13}$$

Minimalisation leads to

$$\frac{\sqrt{n_1^2-1}}{\sqrt{n_2^2-1}} = \frac{\sin\alpha_2}{\sin\alpha_1}\,. \tag{3.14}$$

Indeed this is not Snell's law. The minimisation of proper time does not lead to the correct path of a photon in refractive mediums. In an AEST we will take the action principle in accordance with Fermat's principle. That is, in an AEST the action I of a particle is proportional to the integral of the infinitesimal Euclidean displacement ds. In an AEST the infinitesimal distance ds is c times the time interval dt of the absolute observer. Thus

$$I \propto \int \mathrm{d}s = c\int \mathrm{d}t\,. \tag{3.15}$$

Then we have an action which is valid for both a photon and a massive particle. The proportionality is provided by the Lagrangian $\mathcal{L}$. Thus

$$I = \int \mathcal{L}\mathrm{d}t\,. \tag{3.16}$$

3.3 AEST Lagrangian

To describe kinematics and dynamics in an AEST, we first have to consider the Lagrange formalism in an AEST. In the AEST the observer time t is taken as the parameter, while the proper time of an object is taken as the fourth coordinate. The action in an AEST is therefore given by

$$I = \int_{t1}^{t2} \mathcal{L}(x_\mu(t), u_\mu(t))\mathrm{d}t\,, \tag{3.17}$$

where $\mathcal{L}$ is the Lagrangian in an AEST. As usual, we assume the actual trajectory minimises the action integral. Hence, for the actual motion between t_1 and t_2 the variation of the action δI should be zero, where the variation is such that the endpoints $x_\mu(t_1)$ and $x_\mu(t_2)$ are fixed. For the actual motion the variation of the action should be zero:

$$\delta I = \int_{t1}^{t2} \delta\mathcal{L}\,\mathrm{d}t = 0\,. \tag{3.18}$$

The variation of the integrand yields

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu + \frac{\partial\mathcal{L}}{\partial u_\mu}\delta u_\mu\,. \tag{3.19}$$

Using the relation

$$\delta u_\mu = \frac{\mathrm{d}}{\mathrm{d}t}\delta x_\mu\,, \tag{3.20}$$

we obtain for the variation of the AEST Lagrangian

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial x_\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial u_\mu}\right)\right]\delta x_\mu + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial u_\mu}\delta x_\mu\right]\,. \tag{3.21}$$

In the integration of $\delta\mathcal{L}$ the second term does not contribute because δx_μ is zero at the boundaries. Hence,

$$\int \delta\mathcal{L}\mathrm{d}t = \int \left[\frac{\partial\mathcal{L}}{\partial x_\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial u_\mu}\right)\right]\delta x_\mu \,\mathrm{d}t\,. \tag{3.22}$$

Because δx_μ is arbitrary inside the boundaries, we get the following equations of motion:

$$\frac{\partial\mathcal{L}}{\partial x_\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial u_\mu}\right) = 0\,. \tag{3.23}$$

It is the AEST version of the Euler-Lagrange equations. The derivative of the AEST Lagrangian with respect to the time parameter t is

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x_\mu}{\mathrm{d}t}\right)\frac{\partial\mathcal{L}}{\partial x_\mu} + \left(\frac{\mathrm{d}u_\mu}{\mathrm{d}t}\right)\frac{\partial\mathcal{L}}{\partial u_\mu} = u_\mu\frac{\partial\mathcal{L}}{\partial x_\mu} + \left(\frac{\mathrm{d}u_\mu}{\mathrm{d}t}\right)\frac{\partial\mathcal{L}}{\partial u_\mu}\,. \tag{3.24}$$

Together with the equations of motion (3.23) it leads to

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_\mu\frac{\partial\mathcal{L}}{\partial u_\mu}\right)\,. \tag{3.25}$$

As a consequence we have

$$u_\mu\frac{\partial\mathcal{L}}{\partial u_\mu} - \mathcal{L} = E\,, \tag{3.26}$$

where E is a constant of motion: $\dot{E} = 0$. The quantity E will be referred to as the total energy. It should be distinguished from the energy given by c times the proper time momentum. As we will see further on, the conservation of c times the proper time momentum leads to the conservation of kinetic energy.

By means of the generalised momenta $p_\mu = \partial\mathcal{L}/\partial u_\mu$ the total energy can be regarded as the Hamiltonian:

$$E = \mathcal{H} = p_\mu u_\mu - \mathcal{L}\,. \tag{3.27}$$

3.4 Proper time momentum

The simplest case to consider is the case of a free object in an AEST. The AEST Lagrangian for a free object is

$$\mathcal{L} = m u_\mu u_\mu = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + c^2\dot{\tau}^2)\,. \tag{3.28}$$

The corresponding Euler-Lagrange equations of motion read

$$m u_\mu = p_\mu\,, \tag{3.29}$$

where the momenta p_μ are constants of motion: $\dot{p}_\mu = 0$. Also the mass m of the object is a constant of motion. In the AEST a particular role is played by the proper time momentum $p_4 = mc\dot{\tau}$. Since $u_\mu\dfrac{\partial m u_\mu u_\mu}{\partial u_\mu} = 2m u_\mu u_\mu$ the total energy of a free object is

$$E = m u_\mu u_\mu\,. \tag{3.30}$$

Because $u_\mu u_\mu = c^2$ the total energy can also be written as

$$E = mc^2 \,. \tag{3.31}$$

Since mass is a constant of motion in the AEST theory, we take for it the same value as what is called rest mass in the theory of relativity. In close correspondence with classical mechanics we assume that all momenta will be conserved during an elastic collision. That is

$$\sum_{i=1}^{n} p_{\mu,i} = \sum_{i=1}^{n} p'_{\mu,i} \,. \tag{3.32}$$

The index i identifies the different particles involved in the collision. The prime is used for quantities after the elastic collision. Of special interest is the conservation of proper time momentum:

$$\sum_{i=1}^{n} p_{4,i} = \sum_{i=1}^{n} p'_{4,i} \,, \tag{3.33}$$

or

$$\sum_{i=1}^{n} m_i \sqrt{c^2 - v_i^2} = \sum_{i=1}^{n} m_i \sqrt{c^2 - w_i^2} \,, \tag{3.34}$$

where v_i is the spatial velocity op particle i before the collision, $v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$, and where w_i is the spatial velocity op particle i after the collision, $w_i^2 = \dot{x}'^2_i + \dot{y}'^2_i + \dot{z}'^2_i$. For a classical elastic collision the velocities of objects will be small compared to the speed of light. For $v << c$ the conservation of proper time momentum approximately reads

$$\sum_{i=1}^{n} m_i c \left(1 - \frac{v_i^2}{2c^2}\right) = \sum_{i=1}^{n} m_i c \left(1 - \frac{w_i^2}{2c^2}\right) \,, \tag{3.35}$$

or

$$\sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i w_i^2 \,. \tag{3.36}$$

We see that for $v << c$ the conservation of proper time momentum is reduced to the classical law of conservation of kinetic energy. From the AEST point of view it is better to refer to it as the conservation of proper time momentum. Also the total energy E is conserved. When binding energies are absent, thus when objects are free before and after the collision, the energy conservation reads

$$\sum_{i=1}^{n} m_i c^2 = \sum_{i=1}^{n} m_i c^2 \,. \tag{3.37}$$

The latter is equal to the conservation of mass

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \,. \tag{3.38}$$

3.5 AEST kinematics

In this section we will consider the mechanics of two elastically colliding particles in an AEST [19]. The results will be compared with the SRT. For convenience we confine to a hypothetical head on collision in the x direction. Initially particle 2 is at rest with respect to the absolute restframe while particle 1 moves in the x direction towards particle 2. After the collision the velocity of particle 1 is reduced or even reversed, while particle 2 has obtained a velocity. The situation is shown below.

Figure 3.2: A head on collision of a moving particle against a particle at rest.

Since $v_1 = v$ and $v_2 = 0$ the conservation of momentum in the x direction reads

$$m_1 v = m_1 w_1 + m_2 w_2 \,, \tag{3.39}$$

where w_1 and w_2 are the velocity after the collision of particle 1 and particle 2 respectively. For the conservation of proper time momentum we have

$$m_1\sqrt{c^2 - v^2} + m_2 c = m_1\sqrt{c^2 - w_1^2} + m_2\sqrt{c^2 - w_2^2}\,. \tag{3.40}$$

These are two equations for the two unknowns w_1 and w_2. The non trivial solution is

$$w_1 = \frac{(1-\mu^2)\,v}{1+\mu^2+2\mu\sqrt{1-\frac{v^2}{c^2}}} \tag{3.41}$$

and

$$w_2 = \frac{2\left(\mu+\sqrt{1-\frac{v^2}{c^2}}\right)v}{1+\mu^2+2\mu\sqrt{1-\frac{v^2}{c^2}}}\,, \tag{3.42}$$

where $\mu = \dfrac{m_1}{m_2}$. For various values of the mass ratio μ the velocity w_1 of particle 1 after the collision is plotted against the velocity v of particle 1 before the collision in figure 3.3. For the same mass ratios the velocity w_2 of particle 2 after the collision is plotted against the velocity v of particle 1 before the collision in figure 3.4.

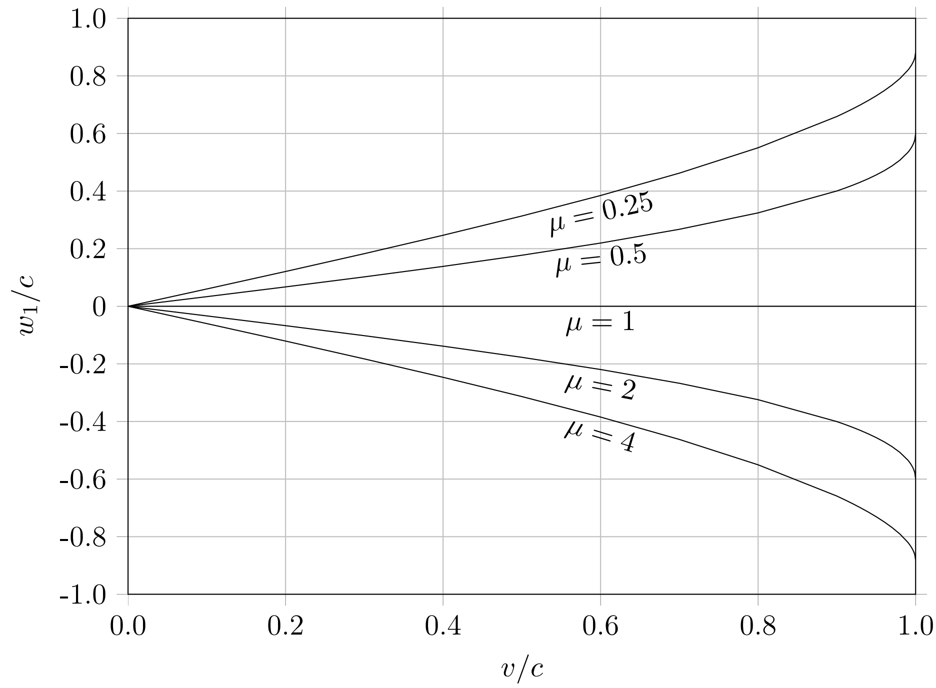

Figure 3.3: A plot of w_1/c against v/c for various mass ratios.

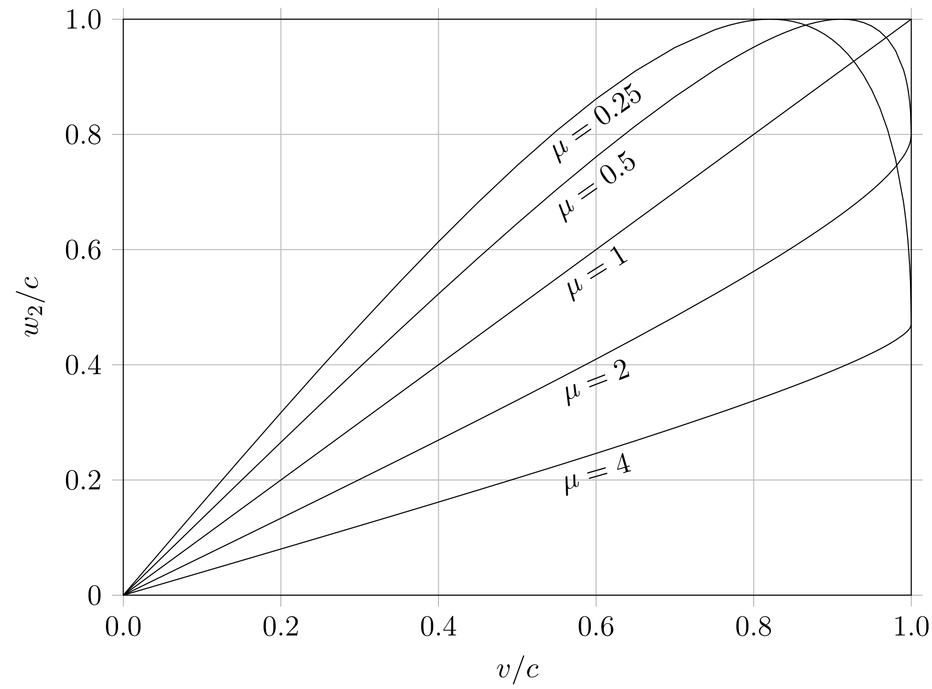

Figure 3.4: A plot of w_2/c against v/c for various mass ratios.

The curves of the AEST predictions for w_2 against v show a symmetry for the change masses $m_1 \leftrightarrow m_2$. That is, if $\mu \to 1/\mu$ then $w_1 \to -w_1$. For w_2 we find that if $\mu \to 1/\mu$ then $w_2 \to w_2 - 2w_1\sqrt{1-\frac{v^2}{c^2}}$.

For the hypothetical situation that $v = c$ there holds $w_1^2 + w_2^2 = c^2$.

More interestingly, for mass ratios smaller than 1 the curves of the AEST predictions for w_2 against v reach a maximum value c when $\frac{v}{c} = \frac{1}{2}\left(\mu + \sqrt{2-\mu^2}\right)$. The smallest value of v for which w_2 obtains such a maximum, occurs in the limit where $\mu \to 0$. That is when $\frac{v}{c} = \frac{1}{2}\sqrt{2}$. Beyond the maximum, thus when $\frac{v}{c} > \frac{1}{2}\left(\mu + \sqrt{2-\mu^2}\right)$, the sign of the proper time of particle 2 after the collision is reversed: $\dot{\tau}_2 < 0$. For the same mass ratios as before the proper time velocity of particle 2 after the collision is plotted against the velocity v of particle 1 before the collision in figure 3.5.

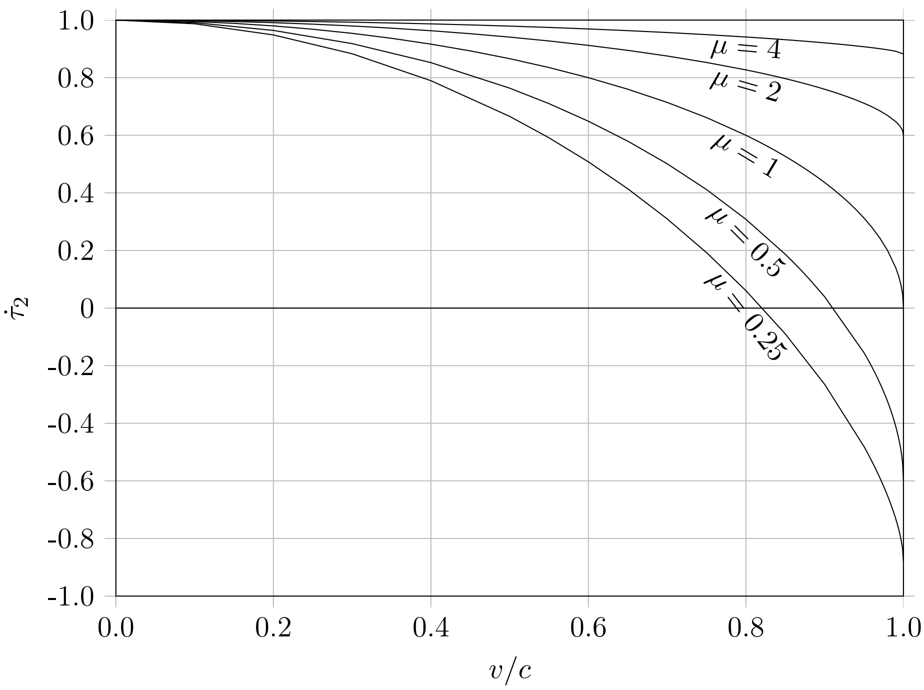

Figure 3.5: A plot of $\dot{\tau}_2$ against v/c for various mass ratios.

Of course, the possibility of proper time reversal can be limited by conservation laws for particle properties. It should also be noted that proper time reversal takes place when the incoming particle has a velocity near the velocity of light, $v > \frac{1}{2}\sqrt{2}$. For such high energies the collision is unlikely to be elastic. Instead, new particles will be created. Nevertheless, the possibility of proper time reversal for particles involved in a collision is an interesting difference with respect to the SRT.

3.6 Comparison with classical mechanics

For $v << c$ equation (3.40) can be approximated by

$$m_1 c\left(1-\frac{v^2}{2c^2}\right)+m_2 c=m_1 c\left(1-\frac{w_1^2}{2c^2}\right)+m_2 c\left(1-\frac{w_2^2}{2c^2}\right) \tag{3.43}$$

or

$$\frac{1}{2}m_1 v^2=\frac{1}{2}m_1 w_1^2+\frac{1}{2}m_2 w_2^2\,. \tag{3.44}$$

We recognise it as the classical law of conservation of kinetic energy. Together with equation (3.39) we have two classical equations of motion for the two unknowns w_1 and w_2. The classical solution is

$$w_1=\frac{1-\mu}{1+\mu}v \tag{3.45}$$

and

$$w_2=\frac{2v}{1+\mu}\,. \tag{3.46}$$

Also here $\mu=\frac{m_1}{m_2}$. Of course, the classical solution can also be obtained by means of the approximation $\sqrt{1-\frac{v^2}{c^2}}\approx 1$ in the AEST predictions for w_1 and w_2.

3.7 Comparison with SRT

In order to make a comparison with the SRT we will give an SRT analysis of the the foregoing particle collision. According to the SRT the conservation of momentum in the x direction and the conservation of energy would read

$$\frac{m_1 v}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_1 w_1}{\sqrt{1-\frac{w_1^2}{c^2}}}+\frac{m_2 w_2}{\sqrt{1-\frac{w_2^2}{c^2}}} \tag{3.47}$$

and

$$\frac{m_1 c^2}{\sqrt{1-\frac{v^2}{c^2}}}+m_2 c^2=\frac{m_1 c^2}{\sqrt{1-\frac{w_1^2}{c^2}}}+\frac{m_2 c^2}{\sqrt{1-\frac{w_2^2}{c^2}}} \tag{3.48}$$

respectively. The non trivial solution for w_1 and w_2 is

$$w_1=\frac{(1-\mu^2)\,v}{1+\mu^2+2\mu\sqrt{1-\frac{v^2}{c^2}}} \tag{3.49}$$

and

$$w_2=\frac{2\left(1+\mu\sqrt{1-\frac{v^2}{c^2}}\right)v}{1+\mu^2+2\mu\sqrt{1-\frac{v^2}{c^2}}+(1-\mu^2)\frac{v^2}{c^2}}\,. \tag{3.50}$$

Again $\mu=\frac{m_1}{m_2}$. We see the SRT prediction for w_1 is identical to the AEST prediction for w_1. The SRT prediction for w_2 differs from the AEST prediction. The difference is negligible for $v << c$. For the same mass ratios as before, the SRT prediction of the velocity w_2 of particle 2 after the collision is plotted against the velocity v of particle 1 before the collision in figure 3.6.

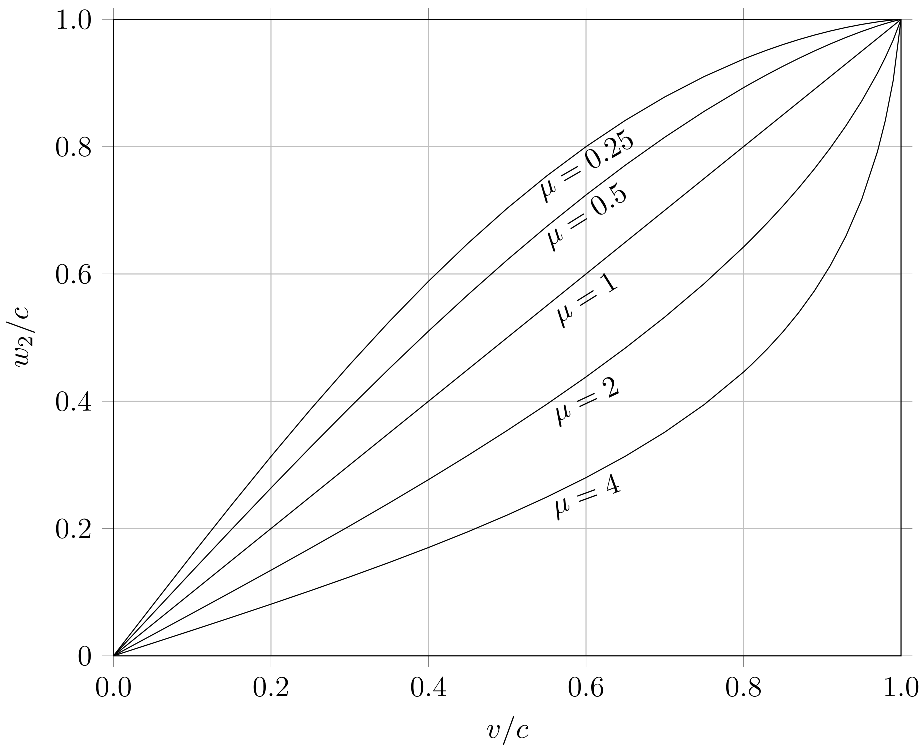

Figure 3.6: A plot of the SRT prediction for w_2/c against v/c for various mass ratios.

3.8 The Compton effect

In the AEST the mass of an object is independent of the velocity of the mass. This allows us to ascribe an intrinsic mass to a photon and a neutrino as follows:

$$mc^2 = hf\,, \tag{3.51}$$

where f is the frequency of an oscillation and where h is Planck's constant. This means that in the AEST the de Broglie relationship between frequency and mass for elementary particles is generalised to photons and neutrinos. For the analysis of the Compton effect explicit use will be made of the mass of the photon [19]. In case of Compton scattering a photon with frequency f is incident on a free electron at rest. On collision, the photon is scattered at an angle θ, while the electron moves off at an angle φ with a velocity $v \neq 0$. Without loss of generality the collision can be considered to take place in the x, y plane. The situation is shown below.

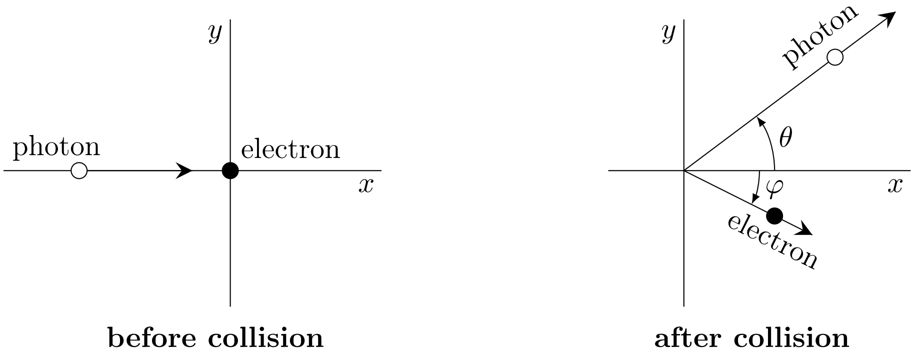

Figure 3.7: A photon colliding against an electron at rest.

In an AEST the conservation of momentum in the x direction and y direction read

$$m_\gamma c = m_{\gamma'} c \cos\theta + m_{e'} v \cos\varphi \tag{3.52}$$

and

$$0 = m_{\gamma'} c \sin\theta - m_{e'} v \sin\varphi \tag{3.53}$$

respectively. The subscripts γ and e identify the photon and electron before the collision and the subscripts γ' and e' identify the photon and electron after the collision. For the conservation of proper time momentum we have

$$m_e c = m_{e'} \sqrt{c^2 - v^2}\,. \tag{3.54}$$

For the conservation of mass we have

$$m_\gamma + m_e = m_{\gamma'} + m_{e'}\,. \tag{3.55}$$

This is a system of four equations for the five unknowns $m_{\gamma'}$, $m_{e'}$, v, θ and φ. Elimination of $m_{e'}$, v and φ leads to

$$m_\gamma m_{\gamma'} (1 - \cos\theta) = m_e (m_\gamma - m_{\gamma'})\,. \tag{3.56}$$

Substituting the equation (3.51) gives

$$h f f' (1 - \cos\theta) = m_e c (f - f')\,, \tag{3.57}$$

where $hf = m_\gamma c^2$ and $hf' = m_{\gamma'} c^2$. Substituting $f = c/\lambda$ and $f' = c/\lambda'$ we arrive at

$$\lambda' - \lambda = \lambda_e (1 - \cos\theta)\,, \tag{3.58}$$

where $\lambda_e = \dfrac{h}{m_e c}$ is the Compton wavelength for the electron. The AEST prediction for the Compton shift is identical to the SRT prediction and (thus) it is in agreement with the experimental value[20] .

The equation (3.54) can be written in the form

$$m_{e'} = \frac{m_e}{\sqrt{1-\frac{v^2}{c^2}}}\,. \tag{3.59}$$

That is, the electron mass has increased during the collision. The price is paid by the photon whose mass (frequency) has decreased during the collision. The relation between the mass of the electron before the collision and the mass of the electron after the collision happens to be precisely the way mass depends on velocity in the SRT. From the AEST point of view it is a result of the mass exchange during the collision.

3.9 Pair annihilation

When an electron and a positron annihilate two photons will emerge. If the electron and positron are moving at the moment of annihilation, the wavelengths of the two photons will be shifted with respect to each other. The shift can be explained in an AEST [19] . We let the electron and positron both move with velocity v in the positive x direction. After the annihilation two photons move off in opposite directions. The situation is shown below.

Figure 3.8: Annihilation process of an electron and a positron.

In an AEST the conservation of momentum in the x direction reads

$$m_e v + m_p v = m_r c - m_l c\,. \tag{3.60}$$

The subscripts e and p identify the electron and the positron. The subscripts r and l identify the photon moving to the right and left respectively. For the conservation of proper time momentum we have

$$m_e\sqrt{c^2-v^2} - m_p\sqrt{c^2-v^2} = 0\,. \tag{3.61}$$

In accordance with the AEST idea that particles and antiparticle have opposite proper time velocities, a negative sign is taken for the proper time velocity of the positron. For the conservation of mass we have

$$m_e + m_p = m_r + m_l\,. \tag{3.62}$$

From the conservation of proper time momentum it follows that the positron and electron must have the same mass, $m_p = m_e$. Then the other two equations are reduced to

$$2m_e v = m_r c - m_l c \tag{3.63}$$

and

$$2m_e = m_r + m_l \,. \tag{3.64}$$

The solution of this simple system is

$$m_r = m_e \left(1 + \frac{v}{c}\right) \tag{3.65}$$

and

$$m_l = m_e \left(1 - \frac{v}{c}\right) \,. \tag{3.66}$$

Hence,

$$\frac{m_l}{m_r} = \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}} \,. \tag{3.67}$$

or

$$\frac{\lambda_r}{\lambda_l} = \frac{c - v}{c + v} \,, \tag{3.68}$$

where λ_r and λ_l are the wavelengths of the photons.

In order to illuminate the similarities and differences between the AEST and the SRT, we will consider four cases in both spacetime models.

Case 1. Both the observer and the electron-positron pair are at rest with respect to the absolute restframe. According to both the AEST and the SRT, there will be no shift: $\dfrac{\lambda_r}{\lambda_l} = 1$.

Case 2. The observer is at rest with respect to the absolute restframe, while the electron-positron pair is moving with a velocity v in the positive x direction. This is just the situation shown in figure 3.8. According to both the AEST and the SRT, the wavelengths will be shifted as given by equation (3.68).

Case 3. The electron-positron pair and the observer are both moving with velocity v in the positive x direction. In the SRT this case is identical to case 1. So, according to the SRT the wavelengths will not be shifted. In an AEST the wavelengths will be shifted. However, the observed values for the wavelengths are also Doppler shifted for an observer moving with respect to the absolute rest frame. The Doppler shift of λ_r and λ_l is

$$\lambda_r' = \lambda_r \sqrt{\frac{c + v}{c - v}} \,, \quad \lambda_l' = \lambda_l \sqrt{\frac{c - v}{c + v}} \,. \tag{3.69}$$

The ratio of the wavelengths, as given by equation (3.68), is precisely compensated by the Doppler shift:

$$\frac{\lambda'_r}{\lambda'_l} = \frac{c+v}{c-v}\frac{\lambda_r}{\lambda_l} = \frac{c+v}{c-v}\frac{c-v}{c+v} = 1\,. \tag{3.70}$$

So, both the AEST and the SRT predict the same in this case.

Case 4. The electron-positron pair is at rest with respect to the absolute restframe, while the observer is moving with velocity v in the negative x direction. According to the SRT there is no difference with Case 2. Therefore, the SRT prediction for the shift will be given by equation (3.68). In an AEST the wavelengths are not shifted since the electron-positron pair is at rest with respect to the absolute restframe. However, the observed values for the wavelengths will be Doppler-shifted for the moving observer. Since the velocity of the observer in this case is opposite to the velocity of case 3, the Doppler shift of λ_r and λ_l is

$$\lambda'_r = \lambda_r\sqrt{\frac{c-v}{c+v}}\,, \quad \lambda'_l = \lambda_l\sqrt{\frac{c+v}{c-v}}\,. \tag{3.71}$$

Since the ratio of the generated wavelengths is 1, the shift is solely caused by the Doppler shift:

$$\frac{\lambda'_r}{\lambda'_l} = \frac{c-v}{c+v}\frac{\lambda_r}{\lambda_l} = \frac{c-v}{c+v}\,1 = \frac{c-v}{c+v}\,. \tag{3.72}$$

So, also in this case both the AEST and the SRT predict the same.
In summary, in all cases both theories predict the same. However, the conceptual differences become very clear. In the AEST theory it is always clear whether the shift is a generated shift due to the velocity of the pair with respect to the absolute restframe, or whether the shift is a Doppler shift due to the motion of the observer with respect to the pair. In the SRT the shift either is a Doppler shift due to the motion of the observer, or a Doppler shift due to the motion of the source, or a combination of both.

3.10 Doppler shift

In a medium the classical Doppler shift is given by

$$f_w = f_s\frac{w+v_w}{w-v_s}\,, \tag{3.73}$$

where w is the velocity of the wave in the medium, f_w is the frequency as observed by the observer, f_s is the source frequency which the source would generate at rest, v_w is the velocity of the observer towards the source and v_s is the velocity of the source towards the observer. In the latter equation time dilation has not been taken into account. If we translate it for the situation in an AEST the quantities would read: w is the velocity of the wave with respect to the absolute restframe, f_w is the frequency as observed by the observer, f_s is the source frequency which the source would generate at rest, v_w is the velocity of the observer with respect to the absolute restframe and v_s is the velocity of the source with respect to the absolute restframe. If an observer is moving with a velocity v_w, its clock will run slower by a factor $\sqrt{1-v_w^2/c^2}$. As a consequence, the observed frequency will be increased by a factor $\frac{1}{\sqrt{1-v_w^2/c^2}}$. If a source is moving with a velocity v_s, its clock will run slower by a factor $\sqrt{1-v_s^2/c^2}$. As a consequence, the generated frequency will be decreased by a factor $\sqrt{1-v_s^2/c^2}$. Correcting the classical Doppler shift for both time dilations, we obtain

$$f_w = \frac{f_s\sqrt{1-v_s^2/c^2}}{\sqrt{1-v_w^2/c^2}} \frac{w+v_w}{w-v_s} \,. \tag{3.74}$$

For classical situations, such as sound waves in air, the time dilation factors are negligible. It becomes of importance if the velocity of the wave is equal to the speed of light. Then the latter equation becomes

$$f_w = f_s \frac{\sqrt{1-v_s^2/c^2}}{\sqrt{1-v_w^2/c^2}} \frac{c+v_w}{c-v_s} = f_s\sqrt{\frac{c+v_s}{c-v_s}}\sqrt{\frac{c+v_w}{c-v_w}} \,. \tag{3.75}$$

If the source is at rest, then

$$f_w = f_s\sqrt{\frac{c+v_w}{c-v_w}} \,. \tag{3.76}$$

If the observer is at rest, then

$$f_w = f_s\sqrt{\frac{c+v_s}{c-v_s}} \,. \tag{3.77}$$

The latter two equations are in correspondence with the shift as it follows from the AEST kinematics. They are also in agreement with the relativistic Doppler shift.
If both v_w and v_s are moving with respect to the absolute restframe the shift is equal to

$$f_w \approx f_s\sqrt{\frac{c+v_w+v_s}{c-v_w-v_s}} + \mathcal{O}\left(\frac{v_w v_s}{c^2}\right) \,. \tag{3.78}$$

Since $v_w + v_b$ is the speed v of the observer with respect to the source, or of the source with respect to the observer, we have

$$f_w \approx f_s\sqrt{\frac{c+v}{c-v}} \,. \tag{3.79}$$

The latter is the well known relativistic Doppler effect.

Chapter 4

Gravitation

4.1 Isotropic metric

In general relativity the Schwarzschild solution for gravitational dynamics in the vicinity of a single spherical source mass reads

$$c^2 \mathrm{d}\tau^2 = (1 - 2\phi)\, c^2 \mathrm{d}t^2 - (1 - 2\phi)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\varphi^2 - r^2 \sin^2 \varphi \mathrm{d}\theta^2 \,, \tag{4.1}$$

where $\phi = \dfrac{GM}{c^2 r}$ is the gravitational potential, with M the mass of the source, and where r is the distance to the center of the gravitational source. From the way the Schwarzschild solution is constructed it should also hold when the source is a spherical shell, a hollow sphere. We consider a sphere with mass M and divide it into a smaller sphere with mass M_1 and a spherical shell around it with mass M_2, see figure 4.1.

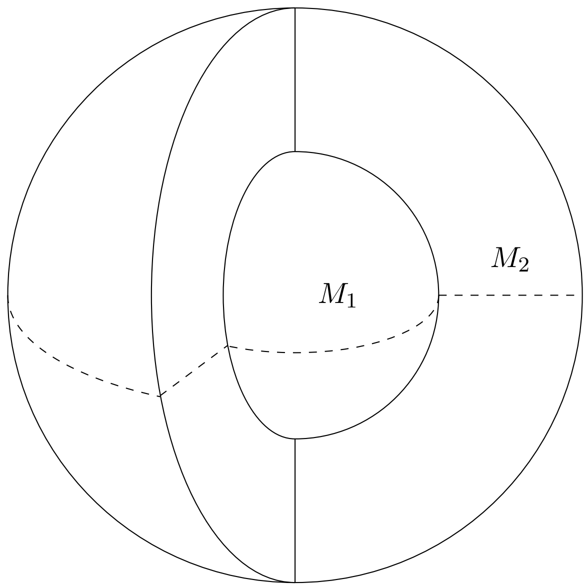

Figure 4.1: A spherical mass divided into an inner sphere and a spherical shell.

The inner sphere corresponds to the metric

$$c^2\mathrm{d}\tau^2 = (1-2\phi_1)\,c^2\mathrm{d}t^2 - (1-2\phi_1)^{-1}\,\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2\,, \tag{4.2}$$

where $\phi_1 = \dfrac{GM_1}{c^2 r}$. The shell around it corresponds to the metric

$$c^2\mathrm{d}\tau^2 = (1-2\phi_2)\,c^2\mathrm{d}t^2 - (1-2\phi_2)^{-1}\,\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2\,, \tag{4.3}$$

where $\phi_2 = \dfrac{GM_2}{c^2 r}$. For the inner sphere and the shell together the metric is given by equation (4.1) with $M = M_1 + M_2$. For merging the inner sphere with the shell the gravitational potentials can simply be added: $\phi = \phi_1 + \phi_2$. For the merging of the metric components we write the three foregoing metrics as

$$c^2\mathrm{d}\tau^2 = g_{tt}(\phi)c^2\mathrm{d}t^2 - g_{rr}(\phi)\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2\,, \tag{4.4}$$

$$c^2\mathrm{d}\tau^2 = g_{tt}(\phi_1)c^2\mathrm{d}t^2 - g_{rr}(\phi_1)\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2 \tag{4.5}$$

and

$$c^2\mathrm{d}\tau^2 = g_{tt}(\phi_2)c^2\mathrm{d}t^2 - g_{rr}(\phi_2)\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2 \tag{4.6}$$

respectively. The relations between the metrical components are

$$g_{tt}(\phi) = g_{tt}(\phi_1) + g_{tt}(\phi_2) - 1\,, \qquad \frac{1}{g_{rr}(\phi)} = \frac{1}{g_{rr}(\phi_1)} + \frac{1}{g_{rr}(\phi_2)} - 1\,. \tag{4.7}$$

Now we consider two separate spheres with mass M_1 and mass M_2, see figure 4.2.

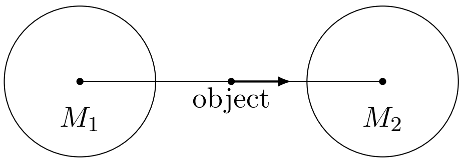

Figure 4.2: Two spherical masses.

The two metrics corresponding to the two masses are

$$c^2\mathrm{d}\tau^2 = (1-2\phi_1)\,c^2\mathrm{d}t^2 - (1-2\phi_1)^{-1}\,\mathrm{d}r_1^2 - r_1^2\mathrm{d}\varphi_1^2 - r_1^2\sin^2\varphi_1\mathrm{d}\theta_1^2 \tag{4.8}$$

and

$$c^2\mathrm{d}\tau^2 = (1-2\phi_2)\,c^2\mathrm{d}t^2 - (1-2\phi_2)^{-1}\,\mathrm{d}r_2^2 - r_2^2\mathrm{d}\varphi_2^2 - r_2^2\sin^2\varphi_2\mathrm{d}\theta_2^2\,, \tag{4.9}$$

where $\phi_1 = GM_1/c^2r_1$ and $\phi_2 = GM_2/c^2r_2$. In general the two metrics cannot simply be merged, because the coordinates $(r_1, \varphi_1, \theta_1)$ with respect to the center of the sphere with mass M_1 differ from the coordinates $(r_2, \varphi_2, \theta_2)$ with respect to the center of the sphere with mass M_2. To avoid this problem, we restrict ourselves to motions along the line connecting the centers of the spheres: $\theta_1 = \theta_2 = 0$ and $\varphi_1 = 0$ and $\varphi_2 = \pi$. For motions along this line

$\mathrm{d}r_1 = -\mathrm{d}r_2$ and thus $\mathrm{d}r_1^2 = \mathrm{d}r_2^2 := \mathrm{d}r^2$. For the restricted motion the metric for the sphere with mass M_1 is reduced to

$$c^2\mathrm{d}\tau^2 = (1 - 2\phi_1)\, c^2\mathrm{d}t^2 - (1 - 2\phi_1)^{-1}\, \mathrm{d}r^2\,, \tag{4.10}$$

while the metric for the sphere with mass M_2 is reduced to

$$c^2\mathrm{d}\tau^2 = (1 - 2\phi_2)\, c^2\mathrm{d}t^2 - (1 - 2\phi_2)^{-1}\, \mathrm{d}r^2\,. \tag{4.11}$$

The latter two metrics can be merged according to equation (4.7). The result is

$$c^2\mathrm{d}\tau^2 = (1 - 2\phi)\, c^2\mathrm{d}t^2 - (1 - 2\phi)^{-1}\, \mathrm{d}r^2\,, \tag{4.12}$$

where $\phi = \phi_1 + \phi_2 = \left(\frac{M_1}{r_1} + \frac{M_2}{r_2}\right)\frac{G}{c^2}$. It leads to an important conclusion: the combined metric for the two spheres may have a singularity even if the separate metrics do not. We give an example. Let $M_1 = M_2 := M$ and let the radii of both spheres be $3GM/c^2$. Then each sphere itself is not a black hole. Now let the distance between the centers be $8GM/c^2$. Precisely halfway between the spheres, $r = 4GM/c^2$, the combined potential is $\phi = 1/2$. At this position $g_{tt} = 0$ and g_{rr} is singular. If we move a little bit towards one of the spheres, to, say, $r_1 = 3.5GM/c^2$ and $r_2 = 4.5GM/c^2$, then $\phi_1 = 1/3.5$, $\phi_2 = 1/4.5$ and $\phi = 1/3.5 + 1/4.5 = 32/63$. Because the potential is larger than $1/2$ at this position, g_{tt} and g_{rr} both have changed sign. So, for motions along the line connecting the two spheres, an object experiences the situation as if it is inside a black hole.

Next we consider four spheres M_1 through M_4. For our purpose we let them be positioned in the $\theta = 0$ plane at equal distances from the barycenter, see figure 4.3.

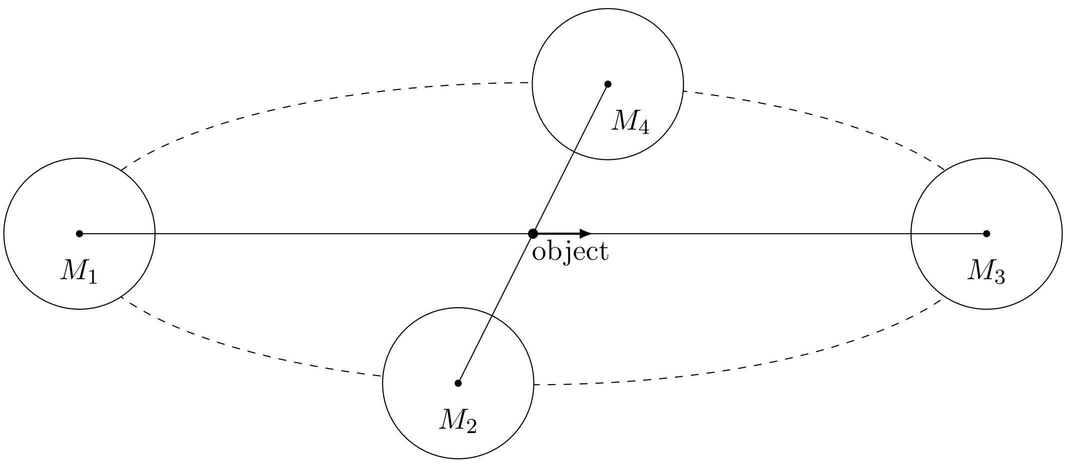

Figure 4.3: Four spherical masses and an object at the barycenter.

If an object is at rest at the barycenter of the four spheres, then all four spheres contribute the same to the decrease of proper time of the object:

$$c^2\mathrm{d}\tau^2 = (1 - 2\phi)\, c^2\mathrm{d}t^2\,, \tag{4.13}$$

where

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 = \frac{G}{c^2} \sum_{i=1}^{4} \frac{M_i}{r_i} \,. \tag{4.14}$$

Again, g_{tt} will be zero when M_i and r_i are such that $\phi = 1/2$. So far there is no problem. A problem arises if one tries to construct the metric for a moving object. For instance, when the object moves along the line connecting M_1 with M_3, see figure 4.3. With respect to M_1, the velocity of the object is in the radial direction. Hence, the potential ϕ_1 contributes to the metric component g_{rr} while it does not contribute to $g_{\varphi\varphi}$. The same holds for M_3. With respect to M_2 and M_4 the velocity is in the tangential φ direction. So, ϕ_2 and ϕ_4 contribute to $g_{\varphi\varphi}$ and not to g_{rr}. If the coordinate system (r, φ, θ) of M_1 is used for the metric, the expression for the interval is

$$c^2 \mathrm{d}\tau^2 = (1 - 2(\phi_1 + \phi_2 + \phi_3 + \phi_4))\, c^2 \mathrm{d}t^2 - (1 - 2(\phi_1 + \phi_3))^{-1} \mathrm{d}r^2 \,. \tag{4.15}$$

Although $g_{\varphi\varphi} = -(1 - 2(\phi_2 + \phi_4))^{-1} r^2$, it does not contribute to the interval since $\mathrm{d}\varphi = 0$. Suppose we let a sphere be build up of smaller spheres as in Apollonian sphere packing, then the potentials of the smaller spheres fully contribute to the potential in g_{tt} of the final sphere, while they partly contribute to the potentials in g_{rr}, $g_{\varphi\varphi}$ and $g_{\theta\theta}$. The only way to get a metric for the larger sphere of identical form as the metric for the smaller spheres, is by using an isotropic metric [21] .

In order to show the potential contributions to g_{tt} we consider the Apollonian sphere packing process for a spherical shell. The outer radius of the shell is R and the inner radius is D, see figure 4.4.

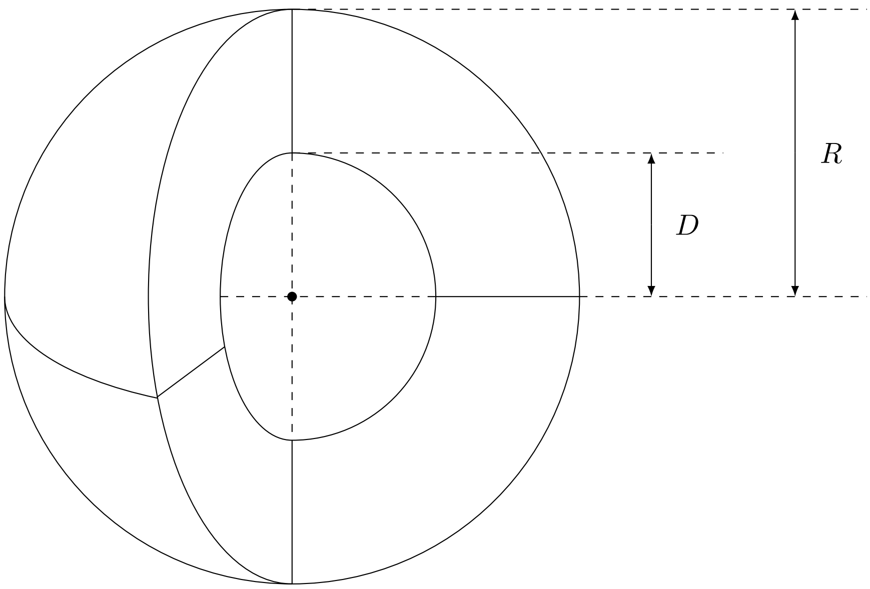

Figure 4.4: A spherical shell with inner radius D and outer radius R.

The inner sphere is empty. The shell is massive. Each part of the shell fully contributes to the potential ϕ_{tt} of the whole shell. Instead of adding the potentials of all the 'Apllonian' parts, it is customary to take the continuum limit and perform the integration.
First we will derive the total potential at the center of empty core. Each infinitesimal volume $\mathrm{d}V = r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$ with mass $\mathrm{d}m = \rho\mathrm{d}V$ will contribute an amount of $\frac{G\mathrm{d}m}{c^2 r}$ to the total potential:

$$\phi_{tt} = \frac{G\rho}{c^2}\int_0^{2\pi}\int_0^{\pi}\int_D^R \frac{r^2\sin\theta}{r}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = 2\pi\rho\frac{G(R^2-D^2)}{c^2}\,. \tag{4.16}$$

Since the mass of a homogeneous shell is given by $M = \frac{4}{3}\pi\rho(R^3 - D^3)$, the latter can also be written as

$$\phi_{tt} = \frac{3GM(R+D)}{2c^2(R^2+RD+D^2)}\,. \tag{4.17}$$

In case of a massive sphere, $D = 0$, the expression for the total potential is reduced to $\phi_{tt} = \frac{3GM}{2c^2R}$, corresponding to the metric component $g_{tt} = \left(1 - \frac{3GM}{c^2R}\right)$. This means that a clock at the center of a massive sphere will stop running if the radius of the massive homogeneous sphere is equal to $3GM/c^2$.
Now we will derive ϕ_{tt} at another position than the center. We call it position A. Without loss of generality we can choose our coordinate system such that A is on the z axis at a distance d from the center: $z_A = d$, $y_A = 0$ and $x_A = 0$. For the distance between A and a point of the shell with coordinates $z = r\cos\theta$, $y = r\sin\theta\sin\varphi$ and $x = r\sin\theta\cos\varphi$ we have

$$l = \sqrt{r^2 - 2rd\cos\theta + d^2}\,. \tag{4.18}$$

The integration over the shell then leads to

$$\phi_{tt} = \frac{G\rho}{c^2}\int_0^{2\pi}\int_0^{\pi}\int_D^R \frac{r^2\sin\theta}{l}\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = \frac{2\pi G\rho}{c^2}\int_0^{\pi}\int_D^R \frac{r^2\sin\theta}{\sqrt{r^2-2rd\cos\theta+d^2}}\,\mathrm{d}r\mathrm{d}\theta\,. \tag{4.19}$$

Now we have to distinguish three situations: point A on the z axis in the empty core, point A on the z axis in outer space outside the shell and point A on the z axis in the shell.

First situation

For the first situation, A on the z axis in the empty core, we have $d < D \le r$. For this situation the integral (4.19) can be written as

$$\phi_{tt} = \frac{2\pi G\rho d^2}{c^2}\int_0^{\pi}\int_{d/R}^{d/D} \frac{\alpha^{-3}\sin\theta}{\sqrt{1-2\alpha\cos\theta+\alpha^2}}\,\mathrm{d}\alpha\mathrm{d}\theta\,, \tag{4.20}$$

where $\alpha = d/r$. By means of the identity

$$\int_0^{\pi} \frac{\sin\theta\,\mathrm{d}\theta}{\sqrt{1-2\alpha\cos\theta+\alpha^2}} = \left.\frac{\sqrt{1-2\alpha\cos\theta+\alpha^2}}{\alpha}\right|_0^{\pi} = 2\,, \tag{4.21}$$

we obtain

$$\phi_{tt} = \frac{4\pi G\rho d^2}{c^2} \int_{d/R}^{d/D} \alpha^{-3}\, \mathrm{d}\alpha = \frac{2\pi G\rho}{c^2}\left(R^2 - D^2\right) . \tag{4.22}$$

Since the mass of a homogeneous shell is given by $M = \frac{4}{3}\pi\rho(R^3 - D^3)$, the latter can also be written as

$$\phi_{tt} = \frac{3GM(R+D)}{2c^2(R^2 + RD + D^2)} . \tag{4.23}$$

We see the result is independent of d. So, ϕ_{tt} is homogeneous inside the empty core. The result therefore is identical to the result (4.17) for the center of the empty core.

Second situation

For the second situation, A on the z axis outside the shell, we have $d > R \geq r$. For this situation the integral (4.19) can be written as

$$\phi_{tt} = \frac{2\pi G\rho d^2}{c^2} \int_0^{\pi} \int_{D/d}^{R/d} \frac{\alpha^2 \sin\theta}{\sqrt{1 - 2\alpha\cos\theta + \alpha^2}}\, \mathrm{d}\alpha \mathrm{d}\theta , \tag{4.24}$$

where $\alpha = r/d$. By means of the identity (4.21) we obtain

$$\phi_{tt} = \frac{4\pi G\rho d^2}{c^2} \int_{D/d}^{R/d} \alpha^2\, \mathrm{d}\alpha = \frac{4\pi G\rho}{3c^2 d}\left(R^3 - D^3\right) . \tag{4.25}$$

With the mass $M = \frac{4}{3}\pi\rho(R^3 - D^3)$ of the homogeneous shell the latter can also be written as

$$\phi_{tt} = \frac{GM}{c^2 d} . \tag{4.26}$$

We see the result is independent of D. The metric component $g_{tt} = 1 - \dfrac{2GM}{c^2 d}$ is in exact agreement with the form of g_{tt} we started with.

Third situation

For the third situation, A on the z axis in the shell, we have $D < d < R$. For the potential contribution of the part of the massive shell with radii larger than d, we can use the result of the first situation and substitute $D = d$. That is,

$$\phi_{tt} = \frac{3GM_1(R+d)}{2c^2(R^2 + Rd + d^2)} , \tag{4.27}$$

where $M_1 = \frac{4\pi}{3}\rho\left(R^3 - d^3\right)$ is the mass of the part of the shell with radii larger than d. For the potential contribution of the part of the massive shell with radii smaller than d, we can use the result of the second situation:

$$\phi_{tt} = \frac{GM_2}{c^2 d} , \tag{4.28}$$

where $M_2 = \frac{4\pi}{3}\rho\left(d^3 - D^3\right)$ is the mass of the part of the shell with radii smaller than d. The total potential is the sum of the contributions. It can be elaborated to

$$\begin{aligned}\phi_{tt} &= \frac{3GM_1(R+d)}{2c^2(R^2+Rd+d^2)} + \frac{GM_2}{c^2 d} = \frac{G}{c^2}\left(\frac{3M_1(R^2-d^2)}{2(R^3-d^3)} + \frac{M_2}{d}\right) \\ &= \frac{G4\pi\rho}{3c^2}\left(\frac{3(R^2-d^2)}{2} + \frac{(d^3-D^3)}{d}\right) = \frac{GM}{c^2}\left(\frac{3(R^2-d^2)}{2(R^3-D^3)} + \frac{d^3-D^3}{d(R^3-D^3)}\right) \\ &= \frac{GM}{c^2}\left(\frac{3d(R^2-d^2)}{2d(R^3-D^3)} + \frac{2(d^3-D^3)}{2d(R^3-D^3)}\right) = \frac{GM}{c^2}\left(\frac{3R^2 d - d^3 - 2D^3}{2d(R^3-D^3)}\right), \end{aligned} \quad (4.29)$$

where $M = M_1 + M_2$. For the metric within a massive sphere, $D = 0$, this is

$$\phi_{tt} = \frac{GM}{c^2}\frac{3R^2 - d^2}{2R^3}\,. \quad (4.30)$$

Since $g_{tt} = 0$ if $\phi_{tt} = 1/2$, we find that A is on the horizon of a black hole if $d = R\sqrt{3 - Rc^2/GM}$. If $R = 3GM/c^2$, the horizon is at $d = 0$, as we saw before. For decreasing R, the horizon of the black hole core will increase. For instance, for $R = 2.99GM/c^2$, the horizon of the black hole is at $d = 0.1R = 0.299GM/c^2$. For $R = 2.64GM/c^2$, the horizon of the black hole is at $d = 0.6R = 1.584GM/c^2$. For $R = 2.36GM/c^2$, the horizon of the black hole is at $d = 0.8R = 1.888GM/c^2$, and so on. For $R = 2GM/c^2$, the horizon of the black hole is at $d = R = 2GM/c^2$. For $R < 2GM/c^2$, the horizon of the black hole stays at $2GM/c^2$, since $\phi_{tt} = \frac{GM}{c^2 d}$ for this case.

The Apollonian sphere packing approach leads to two important conclusions. Firstly, the gravitational potential should be additive; at least in the sense that possible non linear effects are negligible. Secondly, the metric should be isotropic.

4.2 Exponential metric

In this section we will argue that the components of the metric have to be exponential [21]. To this end we once more consider the Schwarzschild solution (4.1). For radial motions it is reduced to

$$c^2\mathrm{d}\tau^2 = (1-2\phi)\,c^2\mathrm{d}t^2 - (1-2\phi)^{-1}\,\mathrm{d}r^2\,. \quad (4.31)$$

For our purpose we rearrange it to

$$c^2\mathrm{d}t^2 = (1-2\phi)^{-1}\,c^2\mathrm{d}\tau^2 + (1-2\phi)^{-2}\,\mathrm{d}r^2\,. \quad (4.32)$$

We can also write the latter as

$$c^2 = (1-2\phi)^{-1}\,c^2\dot{\tau}^2 + (1-2\phi)^{-2}\,\dot{r}^2\,, \quad (4.33)$$

where the dot stands for the derivative with respect to t. In case of a radially moving photon, $\dot{\tau} = 0$, the latter reduces to

$$\dot{r} = (1-2\phi)\,c\,. \quad (4.34)$$

We see that a gravitational potential ϕ decreases the velocity of the photon. The factor $1/(1-2\phi)$ can be interpreted as an index of refraction:

$$\dot{r} = c/n\,, \tag{4.35}$$

where $n = 1/(1-2\phi)$ is the gravitational index of refraction. As in the previous section, we divide the spherical mass M into a spherical core with mass M_1 and a spherical shell with mass M_2, see figure 4.1. The core alone will decrease the radial velocity: $\dot{r}_1 = c/n_1$, where $n_1 = 1/(1-2\phi_1)$. Now, suppose for a moment that the velocity of a free photon is $\dot{r}_1$. Then the shell M_2 alone will reduce the velocity according to the relation $\dot{r} = \dot{r}_1/n_2$, where $n_2 = 1/(1-2\phi_2)$. For the velocity reduction due to the shell it will not matter whether the initial velocity $\dot{r}_1$ of the photon is smaller than c due to an intrinsic property of the photon or due to the gravitational potential of the core. In both cases it is reasonable to expect $\dot{r} = \dot{r}_1/n_2$. So, if the velocity c of the photon is reduced to $\dot{r}_1 = c/n_1$ because of the core M_1, then it is natural to expect that the velocity is further reduced to

$$\dot{r} = \dot{r}_1/n_2 = c/n_1 n_2 \tag{4.36}$$

due to the shell M_2. From the comparison with equation (4.35) we obtain the condition

$$n = n_1 n_2\,. \tag{4.37}$$

The condition is not satisfied if $n = 1/(1-2\phi) = 1/\left(1-2(\phi_1+\phi_2)\right)$, $n_1 = 1/(1-2\phi_1)$ and $n_2 = 1/(1-2\phi_2)$. However, the condition is satisfied if the gravitational refraction index is exponential: $n = \mathrm{e}^{-2\phi} = \mathrm{e}^{-2(\phi_1+\phi_2)}$, $n_1 = \mathrm{e}^{-2\phi_1}$ and $n_2 = \mathrm{e}^{-2\phi_2}$. The difference between $\mathrm{e}^{-2\phi}$ and $1-2\phi$ is of order ϕ^2 and can be neglected. As will be shown further on, the interpretation of the exponential metric as a gravitational index of refraction allows for a geometrical explanation of the bending of light in a flat AEST. Another consequence of the exponential metric is that singularities do not occur. This implies that black holes, in the sense of opposite signs for g_{tt} and g_{rr} inside the Schwarzschild radius, do not exist. However, it does not exclude the possibility of extremely dense and heavy objects in the universe causing black hole like phenomena, such as gravitational waves.

4.3 AEST Lagrangian

In general relativity the Lagrangian for gravitational dynamics is of the form

$$\mathcal{L} = m g_{\mu\nu} u^\mu u^\nu\,, \tag{4.38}$$

where $u_i = \mathrm{d}x_i/\mathrm{d}\tau$ and $u_0 = \mathrm{d}x_0/\mathrm{d}\tau = c\mathrm{d}t/\mathrm{d}\tau$. For a single spherical source mass the Schwarzschild solution reads

$$c^2\mathrm{d}\tau^2 = (1-2\phi)\,c^2\mathrm{d}t^2 - (1-2\phi)^{-1}\,\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2\,, \tag{4.39}$$

where $\phi = \dfrac{GM}{c^2 r}$ is the potential. At this point we wish to make a remark. With the assumption $\mathrm{d}s = c\mathrm{d}\tau$ it appears as if space time is curved:

$$\mathrm{d}s^2 = (1-2\phi)\, c^2\mathrm{d}t^2 - (1-2\phi)^{-1}\,\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2\,. \tag{4.40}$$

However, one could as well have assumed that

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 - r^2\sin^2\varphi\mathrm{d}\theta^2 \tag{4.41}$$

and accept the consequence that $\mathrm{d}s \neq c\mathrm{d}\tau$ in the vicinity of a gravitational source.

In general relativity the isotropic form of the Schwarzschild solution reads

$$c^2\mathrm{d}\tau^2 = \left(\frac{1-\frac{\phi}{2}}{1+\frac{\phi}{2}}\right)^2 c^2\mathrm{d}t^2 - \left(1+\frac{\phi}{2}\right)^4 \left(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + r^2\sin^2\varphi\mathrm{d}\theta^2\right)\,. \tag{4.42}$$

Neglecting terms of order ϕ^2 we can write it in exponential form

$$c^2\mathrm{d}\tau^2 = \mathrm{e}^{-2\phi}c^2\mathrm{d}t^2 - \mathrm{e}^{2\phi}\left(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + r^2\sin^2\varphi\mathrm{d}\theta^2\theta^2\right)\,. \tag{4.43}$$

It can be rearranged to

$$c^2\mathrm{d}t^2 = \mathrm{e}^{2\phi}c^2\mathrm{d}\tau^2 + \mathrm{e}^{4\phi}\left(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + r^2\sin^2\varphi\mathrm{d}\theta^2\right)\,. \tag{4.44}$$

The latter equation is known as Yilmaz' metric [22] . In the present context the latter equation is not a metric. Instead, it is just an equation of motion. According to the AEST the metric ia always $(+1,+1,+1,+1)$. In correspondence with equation (4.44) we take for gravitational dynamics in an AEST the following Lagrangian:

$$\mathcal{L} = m\mathrm{e}^{2\phi}c^2(u_4)^2 + m\mathrm{e}^{4\phi}\left(u_1^2 + u_2^2 + u_3^2\right)\,. \tag{4.45}$$

For this Lagrangian the energy equation

$$E = u_\mu \frac{\partial \mathcal{L}}{\partial u_\mu} - \mathcal{L} \tag{4.46}$$

leads in spherical coordinates to equation (4.44). In an AEST it will not be interpreted as curvature. Distances are still as in a flat Euclidean space:

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,. \tag{4.47}$$

That is, in an AEST $\mathrm{d}s \neq c\mathrm{d}t$ in the vicinity of a gravitational source. In an AEST we rather talk about coefficients $g_{\mu\nu}$ in the Lagrangian, since the $g_{\mu\nu}$ are not components of a metric in an AEST.

In general, the Lagrangian for gravitational motion in an AEST is given by

$$\mathcal{L} = m g_{\mu\nu} u_\mu u_\nu \,, \tag{4.48}$$

where the summation is understood over repeated indices. The Euler-Lagrange equations in an AEST read

$$\frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial u_\mu}\,, \tag{4.49}$$

where μ runs from 1 through 4 and where $x_4 = c\tau$. The energy equation (4.46) leads to

$$mc^2 = m g_{\mu\nu} u_\mu u_\nu \,. \tag{4.50}$$

Again, the latter is just an equation of motion for the four velocities an has nothing to do with curvature. From the Lagrangian (4.48) we obtain

$$\frac{\partial \mathcal{L}}{\partial x_\alpha} = m g_{\mu\nu,\alpha} u_\mu u_\nu \tag{4.51}$$

and

$$\frac{\partial \mathcal{L}}{\partial u_\alpha} = m g_{\alpha\nu} u_\nu + m g_{\mu\alpha} u_\mu \,. \tag{4.52}$$

If g is a symmetrical tensor, one finds for the time derivative of the latter

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial u_\alpha} = m g_{\alpha\nu,\mu} u_\mu u_\nu + m g_{\mu\alpha,\nu} u_\mu u_\nu + 2m g_{\alpha\kappa} \dot{u}_\kappa \,. \tag{4.53}$$

In the latter equations the notation ',α', ',μ' and ',ν' in the subscript stands for the derivative with respect to x_α, x_μ and x_ν. Substitution of the latter results in the Euler-Lagrange equations (4.49) gives

$$a_\kappa + \Gamma_{\kappa\mu\nu} u_\mu u_\nu = 0\,, \tag{4.54}$$

where $a_\kappa = \dot{u}_\kappa = \ddot{x}_\kappa$ and where

$$\Gamma_{\kappa\mu\nu} = \frac{1}{2}\left(g^{-1}\right)_{\kappa\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\,. \tag{4.55}$$

Of course, if g is a symmetrical tensor, the latter could have been written more briefly as $\Gamma_{\kappa\mu\nu} u_\mu u_\nu = \left(g^{-1}\right)_{\kappa\alpha}\left(g_{\mu\alpha,\nu} - \frac{1}{2} g_{\mu\nu,\alpha}\right)$. We have chosen for the form of equation (4.55) just to show the similarity with the Christoffel symbol as it is known in general relativity.

4.4 Gravitational dynamics around a spherical source mass

In case of a spherical source mass M the tensor g is diagonal. As argued in the previous sections, the components of g will be taken exponential and isotropic. To be specific, the components are as given by equation (4.45): $g_{11} = g_{22} = g_{33} = \mathrm{e}^{4\mu/r}$ and $g_{44} = \mathrm{e}^{2\mu/r}$, where $\mu = GM/c^2$ and r is the radial distance of the moving object with respect to the center of the source. For most situations one can restrict to motions in the $z = 0$ plane and thus $\theta = 0$ plane. For these situations the Lagrangian is written in polar coordinates for convenience:

$$\mathcal{L} = m\mathrm{e}^{2\mu/r}(u_4)^2 + m\mathrm{e}^{4\mu/r}\left(\dot{r}^2 + r^2\omega^2\right) , \tag{4.56}$$

where the dot stands for the derivative with respect to time t and where $\omega = \dot{\varphi}$. Now one can first calculate the fields, $\Gamma_{111} = -2\mu r^{-2}$, $\Gamma_{122} = 2\mu - r$, $\Gamma_{144} = \mu r^{-2}\mathrm{e}^{-2\mu/r}$, $\Gamma_{212} = \Gamma_{221} = r^{-1} - 2\mu r^{-2}$, $\Gamma_{414} = \Gamma_{441} = -\mu r^{-2}$ and then derive the equations of motions or one can derive the equations of motions directly from the Lagrangian. Both ways lead to

$$\ddot{r} - \frac{2\mu}{r^2}\dot{r}^2 + 2\mu\omega^2 - r\omega^2 + \frac{\mu}{r^2}\mathrm{e}^{-2\mu/r}(u_4)^2 = 0\,, \tag{4.57}$$

$$\dot{\omega} + \frac{2\dot{r}\omega}{r} - \frac{4\mu\dot{r}\omega}{r^2} = 0\,, \tag{4.58}$$

$$\dot{u}_4 - \frac{2\mu\dot{r}}{r^2}u_4 = 0\,. \tag{4.59}$$

The latter way has the advantage that one directly obtains that $\mathrm{e}^{4\mu r}mr^2\omega$ and $\mathrm{e}^{2\mu r}mu_4$ are constants of motion:

$$\mathrm{e}^{4\mu/r}mr^2\omega = A\,, \tag{4.60}$$

$$\mathrm{e}^{2\mu/r}mu_4 = B\,, \tag{4.61}$$

where A and B are constants of motion [23] . From the energy equation (4.46) we obtain for the Lagrangian (4.56) the equation (4.44) in polar coordinates:

$$c^2\mathrm{d}t^2 = \mathrm{e}^{2\mu/r}c^2\mathrm{d}\tau^2 + \mathrm{e}^{4\mu/r}\left(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2\right) . \tag{4.62}$$

It can also be written as

$$c^2 = \mathrm{e}^{2\mu/r}(u_4)^2 + \mathrm{e}^{4\mu/r}\left(\dot{r}^2 + r^2\omega^2\right) . \tag{4.63}$$

The latter can be rearranged to

$$(u_4)^2 = \mathrm{e}^{-2\mu/r}c^2 - \mathrm{e}^{2\mu/r}v^2\,, \tag{4.64}$$

where $v^2 = \dot{r}^2 + r^2\omega^2$. From the comparison with equation (4.61) we obtain

$$B^2 = \mathrm{e}^{2\mu/r}m^2c^2 - \mathrm{e}^{6\mu/r}m^2v^2\,. \tag{4.65}$$

To first order this is

$$\frac{B^2}{m} = mc^2 + \frac{2GMm}{r} - mv^2 \tag{4.66}$$

or

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \Sigma\,, \tag{4.67}$$

where

$$\Sigma = \frac{m^2c^2 - B^2}{2m} \tag{4.68}$$

is a constant of motion. In equation (4.67) we recognise the classical law of conservation of (kinetic + potential) energy.
Furthermore, in the Newtonian approximation, $\dot{r}^2 + r^2\omega^2 << (u_4)^2 \approx c^2$ and $\mu << r$, the equation (4.57) is reduced to Newton's law:

$$\ddot{r} + GM/r^2 = r\omega^2\,. \tag{4.69}$$

The Newtonian approximation is only of use when both $v << c$ and $\mu << r$.

For a pure circular motion, $\ddot{r} = \dot{r} = 0$, the equations (4.57) and (4.63) are reduced to

$$2\mu\omega^2 - r\omega^2 + \frac{\mu}{r^2}\mathrm{e}^{-2\mu/r}(u_4)^2 = 0\,, \tag{4.70}$$

$$c^2 = \mathrm{e}^{2\mu/r}(u_4)^2 + \mathrm{e}^{4\mu/r}r^2\omega^2\,. \tag{4.71}$$

Eliminiation of u_4 leads to

$$\mu\omega^2 - r\omega^2 + \frac{\mu}{r^2}\mathrm{e}^{-4\mu/r}c^2 = 0 \quad \to \quad r\omega = \sqrt{\frac{\mu}{r-\mu}}\mathrm{e}^{-2\mu/r}c\,. \tag{4.72}$$

The maximum rotational velocity occurs for a photon. Substitution of $u_4 = 0$ into equation (4.71) gives for a photon in a circular orbit

$$r\omega = \mathrm{e}^{-2\mu/r}c\,. \tag{4.73}$$

The substitution of $u_4 = 0$ into equation (4.70) tells us that for a photon $r = 2\mu$. The maximum rotational velocity therefore is $c/\mathrm{e} \approx .367c$. Of course, it can only occur if the radius of the source is smaller than 2μ.

For a pure radial motion, $\omega = 0$, the equations (4.57) and (4.63) are reduced to

$$\ddot{r} - \frac{2\mu}{r^2}\dot{r}^2 + \frac{\mu}{r^2}\mathrm{e}^{-2\mu/r}(u_4)^2 = 0\,, \tag{4.74}$$

$$c^2 = \mathrm{e}^{2\mu/r}(u_4)^2 + \mathrm{e}^{4\mu/r}\dot{r}^2\,. \tag{4.75}$$

Eliminiation of u_4 leads to

$$r^2\ddot{r} - 3\mu\dot{r}^2 + \mu\mathrm{e}^{-4\mu/r}c^2 = 0\,. \tag{4.76}$$

For a pure radial motion of a photon, $\omega = 0$ and $u_4 = 0$, the equations (4.57) and (4.63) are reduced to

$$\ddot{r} - \frac{2\mu}{r^2}\dot{r}^2 = 0\,, \tag{4.77}$$

$$\dot{r} = \mathrm{e}^{-2\mu/r}c\,. \tag{4.78}$$

They are equivalent in that equation (4.77) is the derivative of equation (4.78).

In the next sections we will show that the equations of motion (4.57), (4.60), (4.61) and (4.63) lead to the correct prediction of the deflection of light and the perihelion precession. In the sections thereafter we will derive the orbital precession around two spherical source masses, around a massive disk and around a massive spheroid.

4.5 Gravitational lensing

For the bending of light around a single spherical mass we consider the Euler-Lagrange equation (4.60) and the energy equation (4.63). Since $u_4 = 0$ for a photon, equation (4.63) for a photon reads

$$c^2 = \mathrm{e}^{4\mu/r}\left(\dot{r}^2 + r^2\omega^2\right) . \tag{4.79}$$

By means of the reciprocal radius $u = 1/r$ it can also be written as

$$c^2 = \mathrm{e}^{4\mu u}\left(u'^2 + u^2\right) r^4\omega^2 , \tag{4.80}$$

where the prime denotes the derivative with respect to the orbital angle φ. Substitution of equation (4.60) into equation (4.80) gives

$$c^2 = A^2\mathrm{e}^{-4\mu u}\left(u'^2 + u^2\right) . \tag{4.81}$$

Taking the derivative with respect to φ, we obtain

$$u'' + u = 2\mu\left(u'^2 + u^2\right) . \tag{4.82}$$

To zero order it reads $u'' + u = 0$, which is the equation for a free photon. The zero order solution is $u = \cos\varphi/r_0$, where r_0 is the distance of nearest approach. Substitution of the zero order solution in the right hand side of equation (4.82) gives

$$u'' + u = \frac{2\mu}{r_0^2} . \tag{4.83}$$

The first order solution of this differential equation is

$$u = \frac{\cos\varphi}{r_0} + \frac{2\mu}{r_0^2} . \tag{4.84}$$

At infinity, $u = 0$ and $\varphi = \frac{1}{2}\pi + \varepsilon$, where ε is half the deflection of light. Substitution of these values in the solution (4.84), leads to $\varepsilon = 2\mu/r_0$. Hence, the total deflection of light around a single sphere is given by

$$\Delta\psi = 2\varepsilon = \frac{4\mu}{r_0} . \tag{4.85}$$

4.6 Geometrical explanation of gravitational lensing

In this section it will be shown how the interpretation of the exponential metric as a gravitational index of refraction allows for a geometrical explanation of the bending of light in a flat AEST [24] . Using $v^2 = \dot{r}^2 + r^2\omega^2$ the equation (4.79) can be written as $c = \mathrm{e}^{2\mu/r}v$, where $\mu = GM/c^2$ with M the mass of the source. For the gravitational index of refraction $n = v/c$ we therefore have

$$n = \mathrm{e}^{-2\mu/r}\,, \tag{4.86}$$

That is, for the path of a photon in the gravitational field of a spherical source mass we can use Snell's law. Since the gravitational index of refraction depends on the distance r with respect to the center of the source mass, we have to apply Snell's law everywhere along the path of the photon. We let the path of the photon be in the $z = 0$ plane. Furthermore, we let $t = 0$ be the moment when the photon is at nearest distance r_0 to the source mass at time $t = 0$. The situation is drawn in figure 4.5.

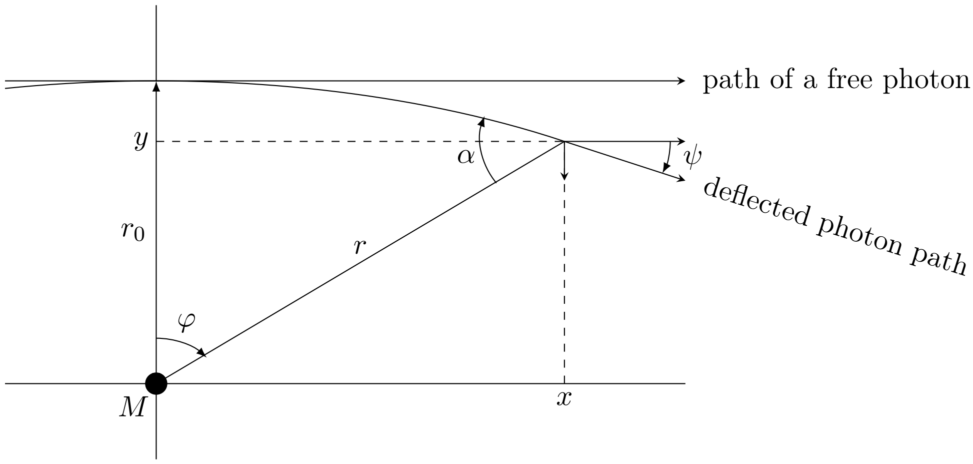

Figure 4.5: Photon path deflected by a gravitational source mass.

For the velocity components and angle α of the deflected photon we have

$$\dot{x} = v\cos\psi\,, \tag{4.87}$$

$$\dot{y} = -v\sin\psi \tag{4.88}$$

and

$$\dot{\alpha} = -\frac{\dot{n}}{n}\tan\alpha - \dot{\phi}\,, \tag{4.89}$$

where v is the velocity of the photon, $x(t)$ and $y(t)$ are the time dependent coordinates of the photon and α is the angle between the direction of the photon and the radius r. The dot represent the derivative with respect to time t. The latter equation might need a little illumination. The radius r is always perpendicular to the circles of equal gravitation. The radius r therefore is perpendicular to the curves of equal index of rrefraction and the angle α

between the photon and the radius r is the incident angle. A naive application of Snell's law would give

$$n(t+\Delta t)\sin\alpha(t+\Delta t) = n(t)\sin\alpha(t)\,. \tag{4.90}$$

The latter is naive in that the angle of incidence $\alpha(t)$ doen not only vary because of the refraction of light, but also because the direction of radius r changes with time. During a time step Δt the radius r is rotated over an angle $\varphi(t+\Delta t)-\varphi(t)$. The equation (4.90) should be modified accordingly:

$$n(t+\Delta t)\sin\left(\alpha(t+\Delta t)+\varphi(t+\Delta t)-\varphi(t)\right) = n(t)\sin\alpha(t)\,. \tag{4.91}$$

In the limit where $\Delta t \to \infty$ one obtains equation (4.89).
For the polar coordinates of the photon we have

$$r = \sqrt{x^2+y^2}\,, \quad \varphi = \arctan(x/y)\,. \tag{4.92}$$

Since we have chosen $t = 0$ for the time of nearest approach, the initial conditions read $x(0) = 0$, $y(0) = r_0$, $\dot{x}(0) = \frac{c}{n(0)}$, $\dot{y}(0) = 0$, $\psi(0) = 0$, $\varphi(0) = 0$ and $\alpha(0) = \frac{\pi}{2}$. With these initial conditions we will solve the set of differential equations (4.87) through (4.89). First we take the time derivative of the polar coordinates:

$$\dot{r} = \frac{x\dot{x}+y\dot{y}}{r} = \dot{x}\sin\varphi + \dot{y}\cos\varphi\,, \tag{4.93}$$

$$r\dot{\varphi} = \frac{y\dot{x}-x\dot{y}}{r} = \dot{x}\cos\varphi - \dot{y}\sin\varphi\,. \tag{4.94}$$

By means of equation (4.87) and equation (4.88) they take the form

$$\dot{r} = v\left(\cos\psi\sin\varphi - \sin\psi\cos\varphi\right)\,, \tag{4.95}$$

$$r\dot{\varphi} = v\left(\cos\psi\cos\varphi + \sin\psi\sin\varphi\right)\,. \tag{4.96}$$

By means of the identity $\alpha = \psi - \varphi + \pi/2$ the equation (4.89) can be written as

$$\dot{\psi} = -\frac{\dot{n}}{n}\tan(\psi-\varphi+\pi/2)\,. \tag{4.97}$$

By means of the trigonometric identity

$$\tan(\psi-\varphi+\pi/2) = \frac{\sin\psi\sin\varphi+\cos\psi\cos\varphi}{\cos\psi\sin\varphi-\sin\psi\cos\varphi}\,, \tag{4.98}$$

we obtain

$$\dot{\psi} = -\frac{\dot{n}}{n}\frac{\sin\psi\sin\varphi+\cos\psi\cos\varphi}{\cos\psi\sin\varphi-\sin\psi\cos\varphi}\,. \tag{4.99}$$

By means of the equations (4.95) and (4.96) we can write

$$\dot{\psi} = -\frac{\dot{n}}{n}\frac{r\dot{\varphi}}{\dot{r}}\,. \tag{4.100}$$

Substitution of equation (4.86) gives

$$\dot{\psi} = \frac{2\mu\dot{\varphi}}{r} \,. \tag{4.101}$$

A zero order solution is the case of a free photon: $x(t) = ct$ and $y(t) = r_0$. For the polar coordinates of the free photon there holds

$$\sin\varphi = \frac{ct}{r}\,, \quad \cos\varphi = \frac{r_0}{r}\,, \quad r^2 = c^2t^2 + r_0^2\,, \quad \varphi = \arctan(ct/r_0)\,, \tag{4.102}$$

and thus

$$\dot{\varphi} = \frac{cr_0}{r^2} \,. \tag{4.103}$$

Substitution of this zero order solution into the right side of equation (4.101) leads to

$$\dot{\psi} = \frac{2\mu c r_0}{r^3} = \frac{2\mu c r_0}{\left(c^2t^2 + r_0^2\right)^{3/2}} \,. \tag{4.104}$$

From the latter differential equation we obtain the deflection of light $\Delta\psi = \psi(\infty) - \psi(-\infty)$ by integration:

$$\Delta\psi = \int_{-\infty}^{\infty} \dot{\psi}\mathrm{d}t = \int_{-\infty}^{\infty} \frac{2\mu c r_0}{\left(c^2t^2 + r_0^2\right)^{3/2}}\mathrm{d}t = \int_{-\infty}^{\infty} \frac{2\mu r_0}{\left(x^2 + r_0^2\right)^{3/2}}\mathrm{d}x \,. \tag{4.105}$$

Performing the standard integral we finally obtain

$$\Delta\psi = \left.\frac{2\mu x}{r_0\left(x^2 + r_0^2\right)^{3/2}}\right|_{-\infty}^{\infty} = \frac{4\mu}{r_0} \,. \tag{4.106}$$

The latter is the correct value for the deflection of light by a spherical source mass.

4.7 Perihelion precession, general part of the analysis

For an unspecified gravitational potential ϕ the Lagrangian (4.45) reads in polar coordinates

$$\mathcal{L} = m\mathrm{e}^{2\phi}(u_4)^2 + m\mathrm{e}^{4\phi}\left(\dot{r}^2 + r^2\omega^2\right) \,. \tag{4.107}$$

The corresponding Euler-Lagrange equations of motion (4.49) are

$$\ddot{r} - \phi_r\,\mathrm{e}^{-2\phi}(u_4)^2 = r\omega^2 + 2\phi_r\left(r^2\omega^2 - \dot{r}^2\right) \,, \tag{4.108}$$

$$\mathrm{e}^{4\mu/r} m r^2 \omega = A \tag{4.109}$$

and

$$\mathrm{e}^{2\mu/r} m u_4 = B \,, \tag{4.110}$$

where ϕ_r stands for the derivative of ϕ with respect to r and where A and B are constants of motion. The energy equation (4.46) reads

$$c^2 = \mathrm{e}^{2\phi}(u_4)^2 + \mathrm{e}^{4\phi}\left(\dot{r}^2 + r^2\omega^2\right) \,. \tag{4.111}$$

For the reciprocal radius $u = 1/r$ there holds for the derivative with respect to the orbital angle φ

$$u' = \frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{-1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \frac{-\dot{r}}{r^2\omega}\,. \tag{4.112}$$

By means of the reciprocal radius the energy equation (4.63) can be written as

$$c^2 = \mathrm{e}^{2\phi}(u_4)^2 + \mathrm{e}^{4\phi}r^4\omega^2\left(u'^2 + u^2\right)\,. \tag{4.113}$$

Substitution of equation (4.109) and equation (4.110) into the latter gives

$$m^2c^2 = \mathrm{e}^{-2\phi}B^2 + \mathrm{e}^{-4\phi}A^2\left(u'^2 + u^2\right)\,. \tag{4.114}$$

Taking the derivative with respect to φ we obtain

$$-\phi_u\mathrm{e}^{-2\phi}B^2 - 2\phi_u\mathrm{e}^{-4\phi}A^2\left(u'^2 + u^2\right) + \mathrm{e}^{-4\phi}A^2\left(u'' + u\right) = 0\,, \tag{4.115}$$

where ϕ_u is the derivative of ϕ with respect to the reciprocal radius. From the latter two equations we obtain

$$-\phi_u\mathrm{e}^{-2\phi}B^2 - 2\phi_u\left(m^2c^2 - \mathrm{e}^{-2\phi}B^2\right) + \mathrm{e}^{-4\phi}A^2\left(u'' + u\right) = 0\,, \tag{4.116}$$

which can elaborated to

$$-r^2\phi_r\mathrm{e}^{-2\phi}A^2\left(u'' + u\right) + B^2 - 2\mathrm{e}^{2\phi}m^2c^2 = 0\,. \tag{4.117}$$

To eliminate B we once more take the derivative with respect to φ. The result is

$$u''' + u'\left(1 + \frac{2r + r^2\phi_{rr}\phi_r^{-1} + 2r^2\phi_r}{r_0(1+e)} - \frac{4r^4\phi_r^2m^2c^2}{\mathrm{e}^{-4\phi}A^2}\right) = 0\,. \tag{4.118}$$

The zero order solution which satisfies $u''' + u' = 0$ is

$$u(\varphi) = \frac{1 + e\cos\varphi}{r_0(1+e)} \quad \text{or} \quad r(\varphi) = \frac{r_0(1+e)}{1 + e\cos\varphi}\,, \tag{4.119}$$

where r_0 is the distance of nearest approach and where e is the eccentricity. For the determination of the constant A we will look for its value at the distance of nearest approach. For $v << c$ and $\mu << r$ the Euler-Lagrange equation (4.108) is reduced to Newton's law for an unspecified potential

$$\ddot{r} - \phi_r c^2 = r\omega^2\,. \tag{4.120}$$

Substituting the zero order solution into the latter equation, we obtain

$$\frac{e\dot{\omega}\sin\varphi + e\omega^2\cos\varphi}{(1 + e\cos\varphi)^2} + \frac{2e^2\omega^2\sin^2\varphi}{(1 + e\cos\varphi)^3} - \frac{\phi_r\,c^2}{r_0(1+e)} = \frac{\omega^2}{1 + e\cos\varphi}\,. \tag{4.121}$$

At the point of nearest approach, $\varphi = 0$, $\dot{r} = 0$, $\dot{\omega} = 0$, $r = r_0$, $\omega = \omega_0$ and $\phi_r = (\phi_r)_0$. Substitution of these values into equation (4.121) leads to

$$r_0^4\,\omega_0^2 = -(\phi_r)_0\,c^2r_0^3(1+e)\,. \tag{4.122}$$

Substitution into equation (4.109) leads, to lowest order, to the following approximation for A:

$$A^2 \approx -(\phi_r)_0\, m^2 c^2 r_0^3 (1+e)\,. \tag{4.123}$$

Substitution of the latter into equation (4.118) leads to

$$u''' + u'\left(1 + \frac{2r + r^2\phi_{rr}\phi_r^{-1} + 2r^2\phi_r}{r_0(1+e)} + \frac{4r^4\phi_r^2}{(\phi_r)_0\, r_0^3(1+e)}\right) = 0\,. \tag{4.124}$$

Since we determined A to lowest order, we have approximated $\mathrm{e}^{-4\phi}$ by 1 for the latter result.

4.8 Orbital precession around a spherical source

For orbits around a sperical source like the Sun, the gravitational potential is $\phi = \mu/r$. Substitution of $\phi_r = -\frac{\mu}{r^2}$, $\phi_{rr} = \frac{2\mu}{r^3}$ and $(\phi_r)_0 = -\frac{\mu}{r_0^2}$ into equation (4.124) leads to [6,23]

$$u''' + u'\left(1 - \frac{6\mu}{r_0(1+e)}\right) = 0\,. \tag{4.125}$$

The solution of the latter differential equation is

$$u(\varphi) = \frac{1 + e\cos\left((1-\chi)\varphi\right)}{r_0(1+e)}\,, \tag{4.126}$$

where χ is

$$\chi = \frac{3\mu}{r_0(1+e)}\,, \tag{4.127}$$

This solution corresponds for an elliptic orbit, $e < 1$, to a precession $\Delta\varphi$ of the perihelion per revolution given by:

$$\Delta\varphi = 2\pi\chi = \frac{6\pi\mu}{r_0(1+e)}\,. \tag{4.128}$$

For an elliptic orbit the semi major axis L is given by $r_0 = L(1-e)$. Then the perihelion precession takes the form

$$\Delta\varphi = \frac{6\pi\mu}{L(1-e^2)}\,. \tag{4.129}$$

4.9 Orbital precession around a bipole mass

Here we analyse the precession for an orbit around two distinct heavy source masses, M_1 and M_2. We let both masses be positioned on the z axis, one at a distance D above the origin and the other at a distance D below the origin. We let an object with relatively small mass m orbit in the $z = 0$ plane. The situation is drawn in figure 4.6.

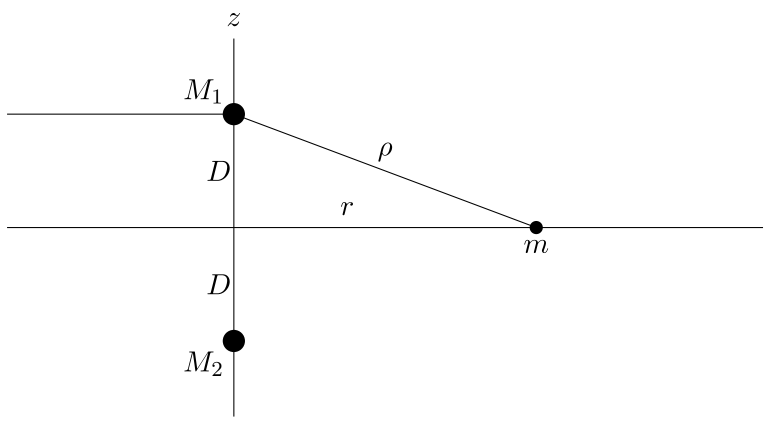

Figure 4.6: Mass m orbitting in the $z = 0$ plane around two source masses on the z axis.

From the geometry we see that the distance ρ between m and M_1, or M_2, and the distance r between m and the z axis are related via $\rho^2 = r^2 + D^2$. In the AEST the total potential is the sum of the potentials of the two source masses:

$$\phi = \frac{G(M_1 + M_2)}{c^2\rho} = \frac{\mu_1 + \mu_2}{\rho} = \frac{\mu}{\sqrt{r^2 + D^2}} \,. \tag{4.130}$$

Throughout this section $\mu = \mu_1 + \mu_2$. Substitution of $\phi_r = -\frac{\mu r}{\rho^3}$, $\phi_{rr} = -\frac{\mu}{\rho^3} + \frac{3\mu r^2}{\rho^5}$ and $(\phi_r)_0 = -\frac{\mu r_0}{\rho_0^3}$ into equation (4.124) leads to

$$u''' + u'\left(1 - \frac{2\mu - 3\rho D^2 u^2}{\rho^3 u^3 r_0(1+e)} - \frac{4\mu\rho_0^3}{\rho^6 u^6 r_0^4(1+e)}\right) = 0\,. \tag{4.131}$$

Since we want the term between brackets to be independent of the angle of the orbit, we will make a series expansion for ρ and ρ_0. For $D << r_0$ we have to first order

$$\rho^n \approx r^n + \frac{nD^2 r^{n-2}}{2} \quad , \quad \rho_0^n \approx r_0^n + \frac{nD^2 r_0^{n-2}}{2} \,. \tag{4.132}$$

To significant order we then obtain

$$u''' + u'\left(1 - \frac{6\mu}{r_0(1+e)} + \frac{3D^2 u}{r_0(1+e)}\right) \approx 0\,. \tag{4.133}$$

For an elliptic orbit with semi major axis L we approximate u by its average value $1/L$. If we also substitute $L(1-e)$ for r_0 into equation (4.133) and neglect terms of insignificant order, we obtain [25]

$$u''' + u'\left(1 - \frac{6\mu - \frac{3\,D^2}{L}}{L(1-e^2)}\right) = 0\,. \tag{4.134}$$

The solution of the latter differential equation is given by equation (4.126), where now χ is

$$\chi = \frac{3\mu - \frac{3D^2}{2L}}{L(1-e^2)}, \tag{4.135}$$

The solution corresponds to a precession $\Delta\varphi$ of the perihelion per revolution given by

$$\Delta\varphi = 2\pi\chi = \frac{6\pi\mu}{L(1-e^2)}\left(1 - \frac{D^2}{2\mu L}\right). \tag{4.136}$$

It should be noted that the latter result is only valid for the situation where $v << c$, $\mu << r$ and $D << L$. In order to get an impression we substitute the values for L and e as they hold for Mercury and the value for μ as it holds for the Sun. The relation between D and the precession $\Delta\varphi$ then is given by

$$\Delta\varphi = 42.9'' \left(1 - \left(\frac{D}{8.7 \cdot 10^{-5}}\right)^2\right), \tag{4.137}$$

where $\Delta\varphi$ is in arcseconds per century and where D is in astronomical units (AU). In figure 4.7 the precession $\Delta\varphi$ is plotted against D.

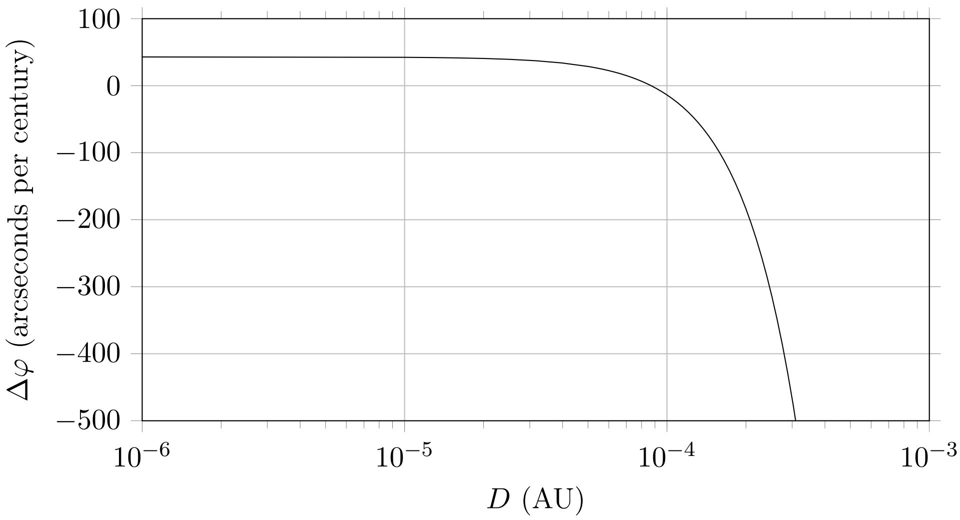

Figure 4.7: The precession of Mercury if it would orbit around a bipole system with the same mass as the Sun.

The bipole system is exactly solved in general relativity [26]. The resulting differential equation has been solved numerically for various values of D [27]. A plot of the numerical general relativity values for the precession against D is in striking agreement with the AEST bipole precession in figure 4.7.

4.10 Orbital precession around a disc

In this section we will analyse the orbital precession for orbits around a disc with a homogeneous mass distribution [25,28] . As for the single spherical source and the bipole, we let an object with mass m orbit in the $z=0$ plane, see the next figure.

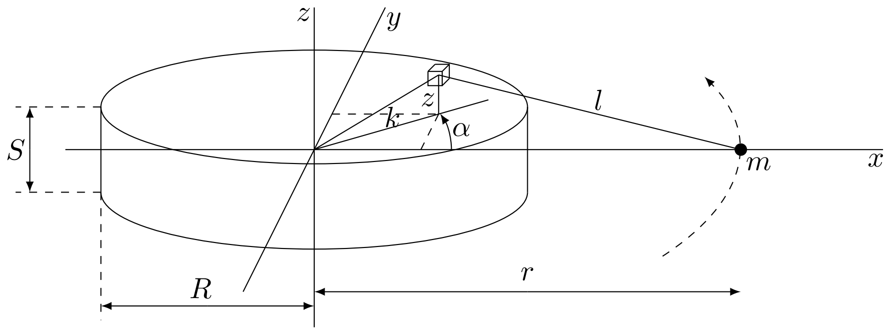

Figure 4.8: An object with mass m orbiting around a disc in the $z=0$ plane.

The disk source has a cylindrical shape with radius R and thickness S. For convenience, the disk is taken in the $z=0$ plane with the center of the disk in the origin. The distance l of m with respect to an infinitesimal volume element in the disk is given by

$$l^2 = (r - k\cos\alpha)^2 + k\sin^2\alpha + z^2 = k^2 - 2kr\cos\alpha + r^2 + z^2\,, \tag{4.138}$$

where (α, z, k) are the cylindrical coordinates in the interior of the disk. To be specific, α is the azimuthal angle, z is the height and k is the radial distance in the $z=0$ plane. For a disk source the potential is given by

$$\phi = \frac{4G\rho}{c^2}\int_0^{S/2}\int_0^{\pi}\int_0^{R} u\,k\,\mathrm{d}k\mathrm{d}\alpha\mathrm{d}z\,, \tag{4.139}$$

where

$$u = \frac{1}{\sqrt{k^2 - 2kr\cos\alpha + r^2 + z^2}} \tag{4.140}$$

is the reciprocal distance of m with respect to an infinitesimal volume element: $u = l^{-1}$. In the disk potential ρ is the homogenous mass density inside the disk. The elliptic integral (4.139) is known as an Epstein-Hubbell integral. The integral can be evaluated exactly by expanding the function u in terms of Legendre polynomials. To this end the function u is written as a Taylor series with respect to k:

$$u = \sum_{n=0}^{\infty}\frac{u_n(0)\,k^n}{n!}\,, \tag{4.141}$$

where $u_n(0)$ stands for the n-th derivative of u with respect to k and evaluated at $k=0$. In addition, a function v is defined as follows: $v = r\cos\varphi - k$. The functions u and v have the following properties: $u_1 = u^3 v$ and $v_1 = -1$. By means of these properties one finds for the higher order derivatives with respect to k:

$$u_n(0) = n!\, w^{-n-1} P_n(w^{-1} r \cos\alpha)\,, \tag{4.142}$$

where $w^2 = r^2 + z^2$ and the P_n are the Legendre polynomials:

$$P_n(x) = \frac{1}{n! 2^n} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^2 - 1\right)^n\,. \tag{4.143}$$

As a consequence, the potential takes the form

$$\phi = \frac{4G\rho}{c^2} \sum_{n=0}^{\infty} \int_0^R k^{n+1} \mathrm{d}k \int_0^{S/2} w^{-n-1} \int_0^{\pi} P_n(w^{-1} r \cos\alpha)\, \mathrm{d}\alpha \mathrm{d}z\,. \tag{4.144}$$

Since the integral vanishes for odd n, the integration over k yields

$$\phi = \frac{4G\rho}{c^2} \sum_{t=0}^{\infty} \frac{1}{2t+2} R^{2t+2} \int_0^{S/2} w^{-2t-1} \int_0^{\pi} P_{2t}(w^{-1} r \cos\alpha)\, \mathrm{d}\alpha \mathrm{d}z\,. \tag{4.145}$$

This integral can be evaluated in a systematic way.[29] For the derivative of the gravitational disk potential with respect to R the result is

$$\mathrm{d}_r \phi = -\frac{GM}{c^2 r^2} \sum_{t=0}^{\infty} A_t B_t(\lambda) C_t(\lambda) \left(\frac{R}{r}\right)^{2t}\,, \tag{4.146}$$

where d_r stands for the derivative with respect to r and where $M = \pi R^2 S \rho$ is the mass of the disk. The coefficients A_t, B_t and C_t are given by

$$A_t = \frac{1}{2^{4t}} \frac{2t+1}{t+1} \binom{2t}{t}^2, \quad B_t(\lambda) = \left(1 + \lambda^2\right)^{-2t-1/2}, \quad C_t(\lambda) = \sum_{i=0}^{t} D_{ti} \lambda^{2i}\,, \tag{4.147}$$

where

$$D_{ti} = \binom{2t}{t}^{-1} \sum_{j=0}^{it} \frac{(-1)^j (2t+2j)!}{(2j+1)!(t+j)!(t-i)!(i-j)!} \tag{4.148}$$

and where $\lambda = \dfrac{S}{2r}$. In case the distance r between the object m and the center of the disk is larger than the size of the disk, either R or S, the derivative of the potential can also be written in an alternative way. To be specific, expanding the $B_t(\lambda)$, writing λ as $\sigma R/r$ and recollecting equal powers of R/r, we obtain

$$\mathrm{d}_r \phi = -\frac{GM}{c^2 r^2} \sum_{t=0}^{\infty} A_t C_{t+1}(\sigma) \left(\frac{R}{r}\right)^{2t}\,, \tag{4.149}$$

where $\sigma = \dfrac{S}{2R}$ is the oblateness of the disk. Explicitly:

$$\begin{aligned} \mathrm{d}_r \phi = & -\frac{GM}{c^2 r^2} \left[1 + \frac{3}{8}\left(1 - \frac{4}{3}\sigma^2\right)\left(\frac{R}{r}\right)^2 + \frac{15}{64}\left(1 - 4\sigma^2 + \frac{8}{5}\sigma^4\right)\left(\frac{R}{r}\right)^4 + \right. \\ & \left. + \frac{175}{1024}\left(1 - 8\sigma^2 + \frac{48}{5}\sigma^4 - \frac{64}{35}\sigma^6\right)\left(\frac{R}{r}\right)^6 + ... \right]. \end{aligned} \tag{4.150}$$

For $R << r$ and $\mu << r$ it suffices to use the following approximations

$$\mathrm{d}_r\phi \approx -\frac{\mu}{r^2}\left[1+\frac{3}{8}\left(1-\frac{4}{3}\sigma^2\right)\left(\frac{R}{r}\right)^2\right], \tag{4.151}$$

$$\mathrm{d}_{rr}\phi \approx \frac{2\mu}{r^3}\left[1+\frac{3}{4}\left(1-\frac{4}{3}\sigma^2\right)\left(\frac{R}{r}\right)^2\right], \tag{4.152}$$

$$(\mathrm{d}_r\phi)^{-1} \approx -\frac{r^2}{\mu}\left[1-\frac{3}{8}\left(1-\frac{4}{3}\sigma^2\right)\left(\frac{R}{r}\right)^2\right]. \tag{4.153}$$

Substituting these expressions and $(\mathrm{d}_r\phi)_0 \approx -\mu/r_0^2$ into equation (4.124), collecting terms of significant order and substituting $L(1-e)$ for r_0 and the average value $1/L$ for $u = 1/r$, we arrive at

$$u''' + u'\left(1-\frac{6\mu + R^2\left(\frac{3}{4}-\sigma^2\right)/L}{L(1-e^2)}\right) \approx 0\,. \tag{4.154}$$

It corresponds to an orbital precession given by

$$\Delta\varphi \approx \frac{6\pi\mu}{L(1-e^2)}\left(1+\frac{R^2\left(\frac{3}{4}-\sigma^2\right)}{6\mu L}\right). \tag{4.155}$$

In order to get an impression we substitute the values for L and e as they hold for Mercury and the value for μ as it holds for the Sun. For each ratio $\sigma = S/2R$, the relation between the disk radius R in AU and the precession $\Delta\varphi$ in arcseconds per century is given by

$$\Delta\varphi \approx 42.9''\left(1+\left(\frac{3}{4}-\sigma^2\right)\left(\frac{R}{1.5\cdot 10^{-4}}\right)^2\right). \tag{4.156}$$

For various σ the orbital precession is plotted against disk radius in figure 4.9.

4.11 Orbital precession around a spheroid

In this section we will analyse the orbital precession for orbits around a spheroid with a homogeneous mass distribution [30]. It will be assumed that the cross section of a spheroid through its center is an ellipse. In analogy with the oblateness of a disk, the oblateness of the spheroid is defined as the semi minor axis divided by the semi major axis of the cross sectional ellipse: $\sigma_{sph} = L_{sph}\sqrt{1-e_{sph}^2}/L_{sph}$ or $\sigma_{sph}^2 = 1-e_{sph}^2$, where e_{sph} is the eccentricity of the ellipse, L_{sph} is the semi major axis of the spheroid and σ_{sph} is the oblateness of the spheroid. We have written the eccentricity and semi major axis of the spheroid as e_{sph} and L_{sph} in order to avoid confusion with the eccentricity e and semi major axis L and of an orbit around the spheroid. We even try to avoid the use of e_{sph} and characterise the oblateness of the spheroid by σ_{sph} as much as possible.

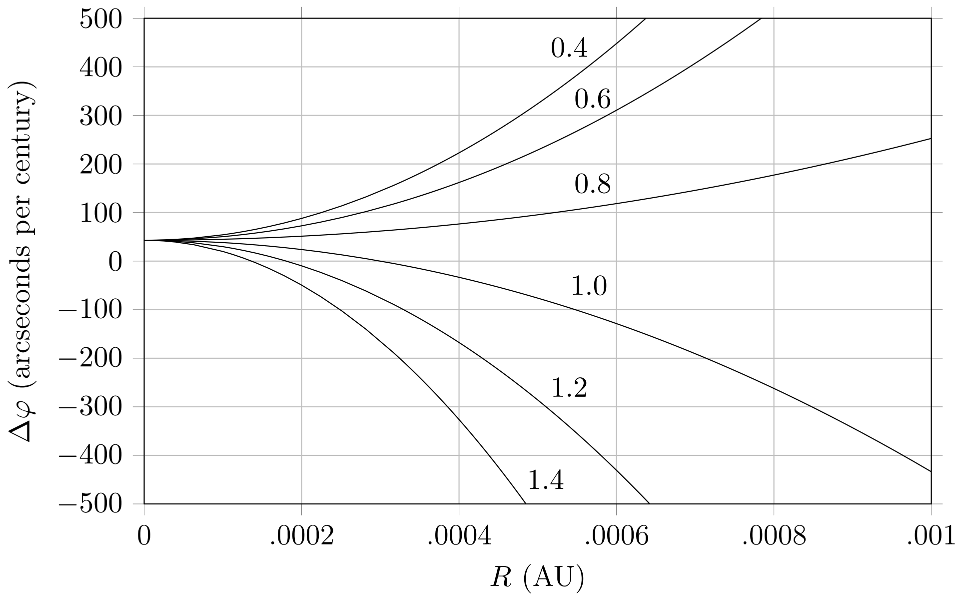

Figure 4.9: The precession of Mercury if it would orbit around a disk with the same mass as the Sun for $\sigma = 0.4$, 0.6, 0.8, 1.0, 1.2 and 1.4.

Now we consider a cylinder with radius q inside the spheroid, $q < L_{sph}$, see the next figure.

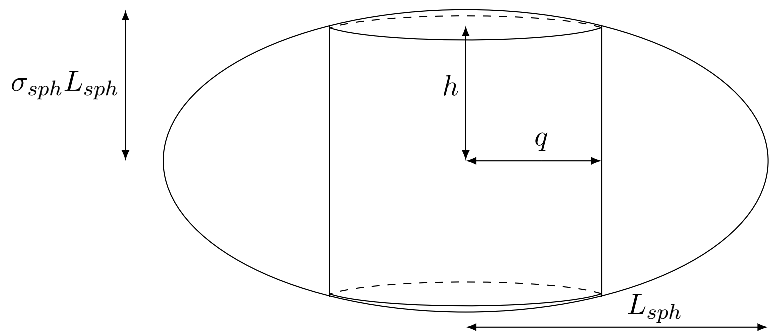

Figure 4.10: A cylinder in the interior of a spheroid.

The cross section through the center then is a rectangle with sizes $2q$ and $2h$, where h and q are related according to the shape of an ellipse: $h = \sigma_{sph}\sqrt{L_{sph}^2 - q^2}$. The height h is with respect to the equatorial plane, thus h is half the height of the cylinder. Next we imagine the spheroid to be build of cylinders. That is, we start with a cylinder with small radius q_0 and height $2h_0 = 2\sigma_{sph}\sqrt{L_{sph}^2 - q_0^2}$ in the middle of the spheroid. Around it one can think a cylindrical tube with inner radius q_0, outer radius q_1 and height $2h_1 = 2\sigma_{sph}\sqrt{L_{sph}^2 - q_1^2}$. Outside that tube is a next tube with inner radius q_1, outer radius q_2 and height $2h_2 = 2\sigma_{sph}\sqrt{L_{sph}^2 - q_2^2}$, and so on until the last tube with outer radius L_{sph} and height equal to 0. The situation is sketched in the next figure.

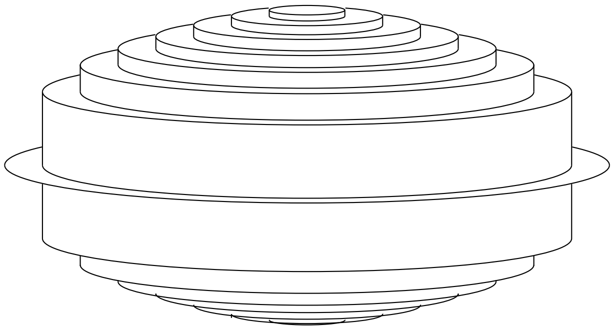

Figure 4.11: An impression of a spheroid divided in cylindrical tubes.

Tube number i with outer radius q_i and height $2h_i = 2\sigma_{sph}\sqrt{L_{sph}^2 - q_i^2}$ can be regarded as a cylinder with radius q_i and height$2h_i$ minus a cylinder with radius q_{i-1} and height $2h_i$. The determination of a potential or some other function, then is a matter of adding up the contributions of all the tubes. For a function F this is:

$$F_{sph} = F_{cyl}(q_0, h_0) + F_{cyl}(q_1, h_1) - F_{cyl}(q_0, h_1) + F_{cyl}(q_2, h_2) - F_{cyl}(q_1, h_2) + +.... + F_{cyl}(q_i, h_i) - F_{cyl}(q_{i-1}, h_i) + ... \tag{4.157}$$

Using

$$F_{cyl}(q_i, h_i) - F_{cyl}(q_{i-1}, h_i) \approx (q_i - q_{i-1})\frac{\partial F_{cyl}}{\partial q} \tag{4.158}$$

and taking the continuum limit, we obtain the following expression for F:

$$F_{sph} = \int_0^{L_{sph}} \left(\frac{\partial F_{cyl}}{\partial q}\right)\Big|_{h(q)} \mathrm{d}q \,. \tag{4.159}$$

To be clear, first we calculate the partial derivative of $F_{cyl}(q, h)$ with respect to q. In the result we substitute the function $h(q)$ for h and perform the integration.
The method will be illustrated by the simple case of the mass of a homogeneous spheroid. For a homogeneous cylinder with density ρ, radius q and height $2h$, the mass is given by $M_{cyl} = 2\pi q^2 h\rho$. The partial derivative with respect to q yields $\frac{\partial M_{cyl}}{\partial q} = 4\pi q h\rho$. Now we substitute $h(q) = \sigma_{sph}\sqrt{L_{sph}^2 - q^2}$ in the integral for the mass of the spheroid:

$$M_{sph} = \int_0^{L_{sph}} (4\pi q h\rho)\Big|_{h(q)} \mathrm{d}q = \int_0^{L_{sph}} 4\pi q\sigma_{sph}\sqrt{L_{sph}^2 - q^2}\rho \mathrm{d}q \,. \tag{4.160}$$

With $x = q/L_{sph}$ the latter can be written as

$$M_{sph} = 4\pi\rho\sigma_{sph}L_{sph}^3 \int_0^1 x\sqrt{1 - x^2}\mathrm{d}x \,. \tag{4.161}$$

By means of the identity $\int_0^1 x\sqrt{1-x^2}\mathrm{d}x = \frac{1}{3}$, we obtain the correct value for the mass of the spheroid:

$$M_{sph} = \frac{4}{3}\pi\rho\sigma_{sph}L_{sph}^3\,. \tag{4.162}$$

Now we turn to the calculation of the derivative of the gravitational potential of a spheroid for distances r larger than the maximum radius of the spheroid; $L_{sph} << r$. By substituting de cylinder mass $2\pi q^2 h\rho$ for M, the height h for $S/2$, the radius q for R and thus h/q for σ in equation (4.150), we obtain for the derivative of the potential of a cylinder in the spheroid:

$$\begin{aligned} \mathrm{d}_r\phi = & -\frac{2\pi\rho G q^2 h}{c^2r^2}\left[1+\frac{3}{8}\left(q^2-\frac{4}{3}h^2\right)\frac{1}{r^2}+\frac{15}{64}\left(q^4-4h^2q^2+\frac{8}{5}h^4\right)\frac{1}{r^4}+\right. \\ & + \left.\frac{175}{1024}\left(q^6-8h^2q^4+\frac{48}{5}h^4q^2-\frac{64}{35}h^6\right)\frac{1}{r^6}+...\right]. \end{aligned} \tag{4.163}$$

The partial derivative with respect to q yields

$$\begin{aligned} \frac{\partial \mathrm{d}_r\phi}{\partial q} = & -\frac{4\pi\rho G q h}{c^2r^2}\left[1+\left(3q^2-2h^2\right)\left(\frac{1}{2r}\right)^2+\left(\frac{45}{4}q^4-30h^2q^2+6h^4\right)\left(\frac{1}{2r}\right)^4+\right. \\ & + \left.\left(\frac{175}{4}q^6-\frac{525}{2}q^4+210h^4q^2-20h^6\right)\left(\frac{1}{2r}\right)^6+...\right]. \end{aligned} \tag{4.164}$$

The substitution of $h(q) = \sigma_{sph}\sqrt{L_{sph}^2-q^2}$ in the integral for the derivative of the potential of the spheroid, followed by the substitution of $M_{sph} = \frac{4}{3}\pi\rho\sigma_{sph}L_{sph}^3$ and $q = xL_{sph}$ leads to

$$\begin{aligned} \left(\frac{\partial \mathrm{d}_r\phi}{\partial q}\right)\Big|_{h(q)} = & -\frac{3GM_{sph}}{c^2r^2L_{sph}}\left[x\sqrt{1-x^2}\left(1-\frac{\sigma_{sph}^2L_{sph}^2}{2r^2}+\frac{3\sigma_{sph}^4L_{sph}^4}{8r^4}-\frac{5\sigma_{sph}^6L_{sph}^6}{16r^6}+...\right)+\right. \\ & + x^3\sqrt{1-x^2}\left(\frac{3L_{sph}^2}{4r^2}+\frac{\sigma_{sph}^2L_{sph}^2}{2r^2}-\frac{15\sigma_{sph}^2L_{sph}^4}{8r^4}-\frac{3\sigma_{sph}^4L_{sph}^4}{4r^4}+...\right)+ \\ & + \left.x^5\sqrt{1-x^2}\left(\frac{45L_{sph}^4}{64r^4}+\frac{15\sigma_{sph}^2L_{sph}^4}{8r^2}+\frac{3\sigma_{sph}^4L_{sph}^4}{8r^2}+...\right)+...\right]. \end{aligned} \tag{4.165}$$

Performing the integration leads to

$$\begin{aligned} \mathrm{d}_r\phi_{sph} = & \int_0^{L_{sph}}\left(\frac{\partial \mathrm{d}_r\phi}{\partial q}\right)\Big|_{h(q)}\mathrm{d}q = L_{sph}\int_0^1\left(\frac{\partial \mathrm{d}_r\phi}{\partial q}\right)\Big|_{h(q)}\mathrm{d}x = \\ & -\frac{3GM_{sph}}{c^2r^2}\left[\frac{1}{3}\left(1-\frac{\sigma_{sph}^2L_{sph}^2}{2r^2}+\frac{3\sigma_{sph}^4L_{sph}^4}{8r^4}-\frac{5\sigma_{sph}^6L_{sph}^6}{16r^6}+...\right)+\right. \\ & + \frac{2}{15}\left(\frac{3L_{sph}^2}{4r^2}+\frac{\sigma_{sph}^2L_{sph}^2}{2r^2}-\frac{15\sigma_{sph}^2L_{sph}^4}{8r^4}-\frac{3\sigma_{sph}^4L_{sph}^4}{4r^4}+...\right)+ \\ & + \left.\frac{8}{105}\left(\frac{45L_{sph}^4}{64r^4}+\frac{15\sigma_{sph}^2L_{sph}^4}{8r^2}+\frac{3\sigma_{sph}^4L_{sph}^4}{8r^2}+...\right)+...\right]. \end{aligned} \tag{4.166}$$

Collecting equal powers of $L_{sph}/2r$ we obtain

$$\begin{aligned} \mathrm{d}_r\phi_{sph} &= -\frac{3GM_{sph}}{c^2r^2}\left[\frac{1}{3}+\frac{2}{5}\left(1-\sigma_{sph}^2\right)\left(\frac{L_{sph}}{2r}\right)^2+\frac{6}{7}\left(1-\sigma_{sph}^2\right)^2\left(\frac{L_{sph}}{2r}\right)^4+\right. \\ &+ \left.\frac{20}{9}\left(1-\sigma_{sph}^2\right)^3\left(\frac{L_{sph}}{2r}\right)^6+\frac{70}{11}\left(1-\sigma_{sph}^2\right)^4\left(\frac{L_{sph}}{2r}\right)^8+...\right]. \end{aligned} \tag{4.167}$$

The latter can also be written as

$$\mathrm{d}_r\phi_{sph} = -\frac{3GM_{sph}}{c^2r^2}\sum_{t=0}^{\infty}\frac{1}{2t+3}\binom{2t}{t}^2\left(\frac{y}{2}\right)^{2t}, \tag{4.168}$$

where y is the ratio of the semi minor axis of the cross section ellipse of the spheroid and the distance r. Thus $y = \frac{L_{sph}}{r}\sqrt{1-\sigma_{sph}^2}$. The latter equation for the potential also is equal to the following power series

$$\mathrm{d}_r\phi_{sph} = -\frac{3GM_{sph}}{2c^2r^2}\left[\frac{\sqrt{1-y^2}}{y^2}-\frac{\arcsin y}{y^3}\right], \tag{4.169}$$

For a perfect sphere, $\sigma_{sph} = 1$, the derivative of the potential is $\mathrm{d}_r\phi_{sph} = -\frac{GM_{sph}}{c^2r^2}$, exactly as required. Since things can no longer be confused with cylinders, we can write R for L_{sph} and leave the subscript *sph* in other quantities of the spheroid source:

$$\mathrm{d}_r\phi = -\frac{3GM}{c^2r^2}\left[\frac{1}{3}+\frac{2}{5}\left(1-\sigma^2\right)\left(\frac{R}{2r}\right)^2+\frac{6}{7}\left(1-\sigma^2\right)^2\left(\frac{R}{2r}\right)^4+...\right]. \tag{4.170}$$

Substitution of the latter expression for $\mathrm{d}_r\phi$ and the approximation $(\mathrm{d}_r\phi)_0 \approx -\mu/r_0^2$ into equation (4.124)leads to

$$u''' + u'\left(1-\frac{K}{L(1-e^2)}\right) \approx 0, \tag{4.171}$$

where K stands for the power series

$$\begin{aligned} K &= 6\mu + \frac{3r+15\mu}{5}(1-\sigma^2)\frac{R^2}{r^2}+\frac{324r+1377\mu}{700}(1-\sigma^2)^2\frac{R^4}{r^4}+ \\ &+ \frac{8184r+29975\mu}{21000}(1-\sigma^2)^3\frac{R^6}{r^6}+\frac{7390528r+23695625\mu}{21560000}(1-\sigma^2)^4\frac{R^8}{r^8}+... \end{aligned} \tag{4.172}$$

For the situation where $\mu << r$ the quantity K is reduced to

$$\begin{aligned} K &= 6\mu + \frac{3r}{5}(1-\sigma^2)\frac{R^2}{r^2}+\frac{81r}{175}(1-\sigma^2)^2\frac{R^4}{r^4}+ \\ &+ \frac{341r}{875}(1-\sigma^2)^3\frac{R^6}{r^6}+\frac{115477r}{336875}(1-\sigma^2)^4\frac{R^8}{r^8}+... \end{aligned} \tag{4.173}$$

With $y = \frac{R}{r}\sqrt{1-\sigma^2}$ it can be written as

$$\begin{aligned} K &= 6\mu + r\left[3\left(\frac{y^2}{5}\right)+\frac{81}{7}\left(\frac{y^2}{5}\right)^2+\frac{341}{7}\left(\frac{y^2}{5}\right)^3+\frac{115477}{539}\left(\frac{y^2}{5}\right)^4+\right. \\ &+ \left.\frac{676661}{7007}\left(\frac{y^2}{5}\right)^5+\frac{59419831}{13377}\left(\frac{y^2}{5}\right)^6+\frac{7380857431}{357357}\left(\frac{y^2}{5}\right)^7+...\right] \end{aligned} \tag{4.174}$$

As can be inferred from equation (4.168) the latter expression for K is equal to the following power series

$$K = 6\mu - r\left[\frac{y^3 - 3y + 3\sqrt{1-y^2}\arcsin y}{y^3 - y + \sqrt{1-y^2}\arcsin y}\right] . \tag{4.175}$$

To order y^2 there approximately holds

$$K \approx 6\mu + \frac{3R^2(1-\sigma^2)}{5r} . \tag{4.176}$$

By means of the average value $1/L$ for $u = 1/r$ this is

$$K \approx 6\mu + \frac{3R^2(1-\sigma^2)}{5L} . \tag{4.177}$$

To the equation

$$u''' + u'\left(1 - \frac{1}{L(1-e^2)}\left(6\mu + \frac{3R^2(1-\sigma^2)}{5L}\right)\right) \approx 0 \tag{4.178}$$

corresponds an orbital precession given by

$$\Delta\varphi \approx \frac{6\pi\mu}{L(1-e^2)}\left(1 + \frac{R^2\left(1-\sigma^2\right)}{10\mu L}\right) . \tag{4.179}$$

Notice that in the latter expression e is the eccentricity of the orbit, while $\sqrt{1-\sigma^2} = e_{sph}$ is the eccentricity of the ellipse cross section of the spheroid. If we substitute the values for L and e as they hold for Mercury and the value for μ as it holds for the Sun, the relation between the spheroid radius R in AU and the precession $\Delta\varphi$ in arcseconds per century is given by

$$\Delta\varphi \approx 42.9''\left(1 + \left(1-\sigma^2\right)\left(\frac{R}{2\cdot 10^{-4}}\right)^2\right) . \tag{4.180}$$

If we also take the radius of the Sun as a fixed value for R, the orbital precession than is given by

$$\Delta\varphi \approx 42.9''\left(1 + 5.5\cdot 10^2\left(1-\sigma^2\right)\right) . \tag{4.181}$$

The latter orbital precession is plotted against σ in the figure 4.12.

The precession reaches a maximum of $2.3\cdot 10^4$ arcseconds per century when $\sigma = 0$. That is, if the Sun would be an almost flat disk with the same radius as the radius of the Sun, Mercury would orbit around it with a precession of almost 6.5 degrees per century. The curve intersects the horizontal $\Delta\varphi = 0$ line at $\sigma \approx 1.00092$. For the Sun $\sigma \approx 0.999991$, which corresponds to a precession of 43.3 arcseconds per century, which is about 1% larger than the 42.9 arcseconds per century it would be if the Sun was a perfect sphere. The situation for σ close to 1 is shown in the figure 4.13.

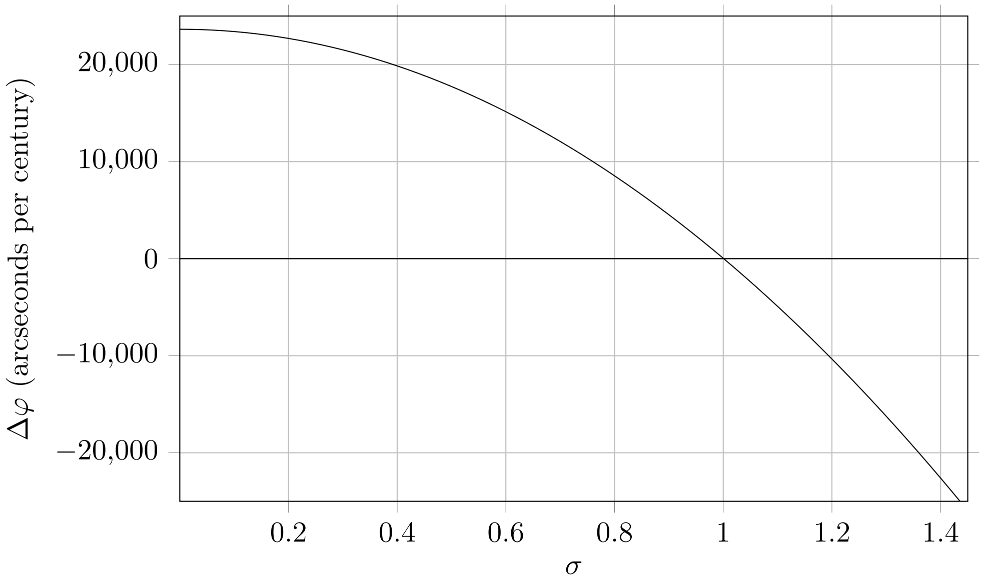

Figure 4.12: The precession of Mercury if it would orbit around a spheroid with the same mass as the Sun and with an orbital radius equal to the one of Mercury., plotted against σ.

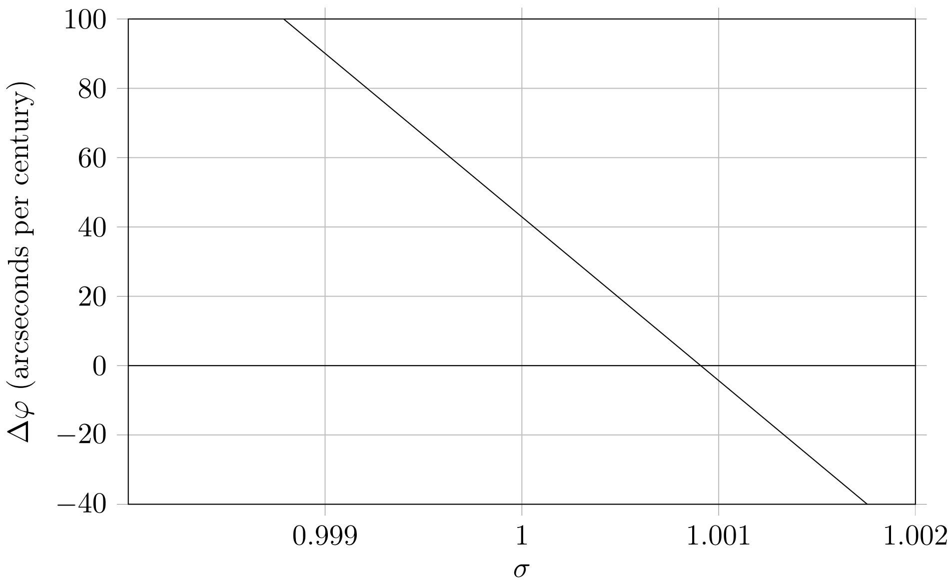

Figure 4.13: The precession of Mercury if it would orbit around a almost perfect sphere with the same mass as the Sun and with an orbital radius equal to the one of Mercury, plotted against σ.

Chapter 5

Electrodynamics

5.1 Generators

The four acceleration a_μ of an object with respect to the absolute restframe is the derivative of the four velocity v_μ with respect to the time parameter t:

$$a_\mu = \dot{v}_\mu = \ddot{x}_\mu \,. \tag{5.1}$$

The four force K_μ of an object is mass times the four acceleration:

$$K_\mu = m a_\mu \,. \tag{5.2}$$

For a motion in the $x, c\tau$ plane the change of the two velocity components caused by the rotation of the velocity over an angle φ_{14} is shown in the next figure.

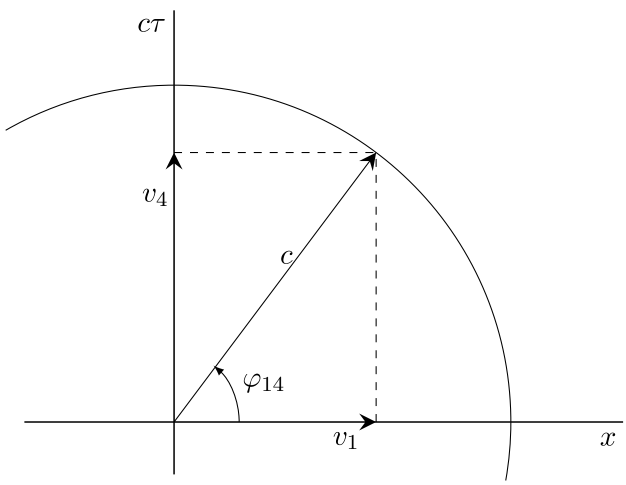

Figure 5.1: AEST velocity diagram.

For the two velocity components we have

$$\begin{pmatrix} v_1 \\ v_4 \end{pmatrix} = \begin{pmatrix} c\cos\varphi_{14} \\ c\sin\varphi_{14} \end{pmatrix} \tag{5.3}$$

The derivative of the velocity components with respect to t gives

$$\begin{pmatrix} a_1 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_4 \end{pmatrix} \omega_{14}\,, \tag{5.4}$$

where the angular velocity is given by $\omega_{14} = \dot{\varphi}_{14}$. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the infinitesimal generator of the SO(2) group. It generates a boost in the x_1 direction. Accelerations in a four dimensional AEST are covered by the six dimensional group SO(4). The infinitesimal generators of the group SO(4) are

$$M_{12} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{13} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$M_{14} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad M_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_{34} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

The generators M_{12}, M_{13} and M_{23} correspond to rotations in the spatial dimensions. The generators M_{14}, M_{24} and M_{34} correspond to boosts in the directions x_1, x_2 and x_3. For $M_{\mu\nu}$ there holds $M_{\nu\mu} = -M_{\mu\nu}$ and $M_{\mu\mu} = 0$. An element of $M_{\mu\nu}$ with row index ρ and column index σ will be denoted as $M_{\mu\nu\rho\sigma}$. There holds

$$M_{\mu\nu\rho\sigma} = \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}\,. \tag{5.6}$$

As a consequence,

$$M_{\mu\nu\rho\sigma} = -M_{\nu\mu\rho\sigma} \quad , \qquad M_{\mu\nu\rho\sigma} = -M_{\mu\nu\sigma\rho} \quad , \qquad M_{\mu\nu\rho\sigma} = M_{\rho\sigma\mu\nu}\,. \tag{5.7}$$

The commutation relations for $M_{\mu\nu}$ can be summarised in the following analogue of the Lorentz algebra

$$[M_{\mu\nu}, M_{\kappa\lambda}] = \delta_{\mu\kappa}M_{\nu\lambda} - \delta_{\mu\lambda}M_{\nu\kappa} - \delta_{\nu\kappa}M_{\mu\lambda} + \delta_{\nu\lambda}M_{\mu\kappa}\,. \tag{5.8}$$

The elements of the Lie group SO(4) can be written in exponential form. A change of the four velocity can therefore be written as

$$v'_\rho = v_\sigma \mathrm{e}^{\frac{1}{2}\varphi_{\mu\nu}M_{\mu\nu\rho\sigma}}\,. \tag{5.9}$$

The factor 1/2 appears because for φ_{13}, for instance, both $\varphi_{13}M_{13}$ and $\varphi_{31}M_{31}$ contribute. For an infinitesimal change $\Delta\varphi_{\mu\nu} = \omega_{\nu\mu}\Delta t$, the equation (5.9) leads to

$$v_\rho(t+\Delta t) = v_\rho(t) + \frac{1}{2}v_\sigma(t)\omega_{\mu\nu}M_{\mu\nu\rho\sigma}\Delta t + \mathcal{O}\left((\Delta t)^2\right) . \tag{5.10}$$

From the latter we obtain the following expression for the acceleration:

$$a_\rho(t) = \lim_{\Delta t\to 0}\frac{v_\rho(t+\Delta t) - v_\rho(t)}{\Delta t} = \frac{1}{2}v_\sigma(t)\omega_{\mu\nu}M_{\mu\nu\rho\sigma} , \tag{5.11}$$

or

$$a_\rho = \frac{1}{2}v_\sigma\omega_{\mu\nu}M_{\mu\nu\rho\sigma} . \tag{5.12}$$

Substitution of equation (5.6) into equation (5.12) results in

$$a_\rho = v_\sigma\omega_{\sigma\rho} . \tag{5.13}$$

Since $\omega_{\mu\nu}$ is antisymmetric, it follows that $a_\mu v_\mu = 0$, as required. For dynamics in an AEST we let the $\omega_{\mu\nu}$ be caused by fields $F_{\nu\mu}$. If, for instance, $\omega_{\mu\nu} = \frac{q}{m}F_{\nu\mu}$, then the four force reads

$$K_\mu = ma_\mu = qv_\nu F_{\mu\nu} . \tag{5.14}$$

In the next section we will investigate the latter force law in case of electrodynamic fields.

5.2 AEST electrodynamics

The AEST Lagrangian for the motion of a charged particle in an electromagnetic field is [31]

$$\mathcal{L} = mv_\mu v_\mu + 2qA_\mu v_\mu , \tag{5.15}$$

where q is the charge of the object and where A_μ is the electromagnetic potential field. Since mass is a constant of motion in the AEST, the presence of m in the Lagrangian (5.15) does not cause disturbing derivatives of mass with respect to velocity in the equations of motions. The Lagrangian (5.15) preserves $E = mc^2$. Indeed, for the Lagrangian (5.15), equation (3.26) gives

$$E = mv_\mu v_\mu = mc^2 . \tag{5.16}$$

For the Lagrangian (5.15) the equations of motion (3.23) result in

$$K_\mu = ma_\mu = q\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)v_\nu . \tag{5.17}$$

The partial derivatives will be taken with respect to an observer at rest in the preferred frame. That is, $\partial_i = \partial/\partial x_i$ and $\partial_4 = c^{-1}\partial/\partial t$. Comparison of equation (5.14) with equation (5.17) leads to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{5.18}$$

The electric and magnetic field will be defined as

$$E_i = cF_{i4} \leftrightarrow F_{i4} = \frac{E_i}{c} \tag{5.19}$$

and

$$B_k = \frac{1}{2}\varepsilon_{ijk}F_{ij} \leftrightarrow F_{ij} = \varepsilon_{ijk}B_k \tag{5.20}$$

respectively, where ε is the Levi Civita tensor. That is

$$F = \begin{pmatrix} 0 & B_3 & -B_2 & E_1/c \\ -B_3 & 0 & B_1 & E_2/c \\ B_2 & -B_1 & 0 & E_3/c \\ -E_1/c & -E_2/c & -E_3/c & 0 \end{pmatrix} . \tag{5.21}$$

With these definitions for the electric and magnetic field the four force takes the form

$$K_4 = qv_jF_{j4} = -qv_j\frac{E_j}{c} \tag{5.22}$$

and

$$K_i = qv_\nu F_{i\nu} = qv_4F_{i4} + qv_jF_{ij} = q\left(v_4\frac{E_i}{c} + \varepsilon_{ijk}\, v_jB_k\right) . \tag{5.23}$$

In vector notation the latter reads

$$\vec{K} = q\left(s\frac{\vec{E}}{\gamma} + \vec{v}\times\vec{B}\right) , \tag{5.24}$$

where s is the sign of proper time. The latter equation is the AEST analogue of the Lorentz force. It differs by a factor γ and a sign s from the Lorentz force as it reads in the SRT:

$$\vec{K} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) . \tag{5.25}$$

The reason for the factor $1/\gamma$ in the AEST version of the Lorentz force is obvious. In the SRT the time parameter is taken as the fourth coordinate. The velocity of time will therefore be c in the SRT. In the AEST the proper time of an object is taken as the fourth coordinate. The absolute value of the proper time velocity will therefore be equal to $|v_4| = c/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$ with v the spatial velocity of the object which experiences the electromagnetic field. To illuminate a consequence, we consider a charged particle subject to a linear acceleration in a pure electric field. The AEST analogue of the Lorentz force then reads $m_0\vec{a} = q\vec{E}/\gamma$, while in the SRT the Lorentz force reads $m_0\gamma\vec{a} = q\vec{E}$. Both expressions are mathematically identical. Conceptually they are different. In the AEST the factor γ is due to sensitivity to an electric field being proportional to the proper time velocity. In particular a particle with reversed proper time velocity, an antiparticle, will respond oppositely to an electric field. However, its response to a magnetic field will not be reversed.

5.3 Coulomb's law

The proper time velocity has consequences for Coulomb's law. In classical electrodynamics the electric force between two charges is given by Coulomb's law:

$$K = \frac{q_1\, q_2}{4\pi\varepsilon_0 r^2}\,, \tag{5.26}$$

where r is the distance between the charges q_1 and q_2 and where ε_0 is the vacuum permittivity. In the AEST version of the Lorentz force the sensitivity of a charge q_1 to an electric field will be proportional to its proper time velocity $\dot{\tau}_1$. Similarly, in the AEST the electric field generated by a charge q_2 is proportional to the proper time velocity $\dot{\tau}_2$ of charge q_2. It is comparable to the sensitivity of a moving charge to a magnetic field being proportional with its spatial velocity, while the intensity of the magnetic field generated by a moving charge is proportional to the velocity of the source charge. To be specific, according to the AEST version of Coulomb's law is given by

$$K = \frac{q_1\dot{\tau}_1\, q_2\dot{\tau}_2}{4\pi\varepsilon_0 r^2}\,. \tag{5.27}$$

It can also be written as

$$K = \frac{s_1 s_2 q_1\, q_2}{4\pi\varepsilon_0 r^2\gamma_1\gamma_2}\,, \tag{5.28}$$

where the index 1 refers to the charge which experiences the electric field and where the index 2 refers to the charge which generates the electric field. The s_1 and s_2 are the signs of the proper time of charges q_1 and q_2 respectively. If both charges have the same sign of proper time the AEST version of Coulomb's law reduces to

$$K = \frac{q_1\, q_2}{4\pi\varepsilon_0 r^2\gamma_1\gamma_2} = \frac{q_1\sqrt{c^2-v^2}\, q_2\sqrt{c^2-u^2}}{4\pi\varepsilon_0 c^2 r^2}\,. \tag{5.29}$$

where v is the velocity of the charge which experiences the electric field and where u is the velocity of the charge which generates the electric field. Symmetry requires that both proper time velocities should be present in the electric force. If one of the proper time velocities is left, Newton's action-equals-reaction principle would not be satisfied. Only if both charges q_1 and q_2 are at rest with respect to the absolute rest frame and have the same sign of proper time, the AEST version of Coulomb's law is identical to the SRT version of Coulomb's law.

5.4 Hydrogenic atoms

For the explanation of the spectra of hydrogenic atoms in an AEST, we have to consider the bound orbits of an electron in the vicinity of a charged nucleus. For our purpose it is sufficient to consider the fourth component, the electric part, of the potential field. Then the Lagrangian for the motion of the electron reads

$$\mathcal{L} = m v_\mu v_\mu - 2eA_4 v_4\,. \tag{5.30}$$

If the source of the electric potential is an elementary particle at rest, then A_4 would be given by $A_4 = \dfrac{-Ze}{4\pi\varepsilon_0 rc}$. However, in case of a hydrogenic atom the electric potential is generated by a nucleus composed of quarks. The quarks inside the nucleus will have a motion. As a consequence the sum of all the quark charges times the quark proper time velocities will be smaller than Zec. Let us denote it as $Zec\,\xi$, where $0 < \xi < 1$. Then

$$A_4 = \frac{-Ze\xi}{4\pi\varepsilon_0 rc} \tag{5.31}$$

and

$$\mathcal{L} = mv_\mu v_\mu + \frac{2Ze^2\xi\, v_4}{4\pi\varepsilon_0 rc}\,. \tag{5.32}$$

If we convert to polar coordinates θ and ϕ and confine to a motion on the $\theta = 0$ plane, the Lagrangian reads

$$\mathcal{L} = m\left(\dot{r}^2 + r^2\omega^2\right) + m(v_4)^2 + \frac{2Ze^2\xi\, v_4}{4\pi\varepsilon_0 rc}\,, \tag{5.33}$$

where $\omega = \dot{\phi}$. For this AEST Lagrangian the equations of motion read

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} \qquad \to \qquad m\ddot{r} = mr\omega^2 - \frac{Ze^2\xi\, v_4}{4\pi\varepsilon_0 r^2 c}\,, \tag{5.34}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\omega} \qquad \to \qquad mr^2\omega = L \tag{5.35}$$

and

$$\frac{\partial\mathcal{L}}{\partial x_4} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial v_4} \qquad \to \qquad mv_4 + \frac{Ze^2\xi}{4\pi\varepsilon_0 rc} = B\,, \tag{5.36}$$

where L and B are constants of motion: $\dot{L} = 0$ and $\dot{B} = 0$. Of course, L is the angular momentum of the electron. As we will see, B is related to the energy or classical Hamiltonian. With the assumption that the sign of v_4 is positive, the equation for B can also be written as

$$mc\sqrt{c^2 - v^2} + \frac{Ze^2\xi}{4\pi\varepsilon_0 r} = Bc\,, \tag{5.37}$$

where v is the spatial velocity: $v = \sqrt{v_k v_k}$. With the approximation $mc\sqrt{c^2 - v^2} \approx mc^2 - \frac{1}{2}mv^2$ it is reduced to

$$\frac{1}{2}mv^2 - \frac{Ze^2\xi}{4\pi\varepsilon_0 r} = \Sigma\,, \tag{5.38}$$

where $\Sigma = mc^2 - Bc$ is a constant of motion. If $\xi = 1$, the equation (5.38) is the conservation of the sum of classical kinetic energy and potential energy. Therefore, the constant Σ can be regarded as the classical Hamiltonian. The constants of motion B and Σ will depend on the initial velocity of the electron. As can be inferred form equation (5.34), for a slow circular orbit of the electron

$$\frac{1}{2}mv^2 \approx \frac{Ze^2\xi}{8\pi\varepsilon_0 r}\,. \tag{5.39}$$

As a consequence, for a slow orbit

$$\Sigma \approx -\frac{Ze^2\xi}{8\pi\varepsilon_0 r} \,. \tag{5.40}$$

If $\xi = 1$, the equations (5.34), (5.35) and (5.39) are the classical equations of motion. We could therefore proceed in the classical way. Yet, there is a substantial difference. In Bohr's model the electron absorbs or emits the energy of a photon in order to make a transition to a state of different energy, while in an AEST the electron absorbs or emits the mass of the photon in order to make a transition to another state. Because of the conceptual difference an analysis will be given on the basis of the equations (5.34), (5.35) and (5.36). We start with equation (5.36). Multiplication of equation (5.36) with the proper time velocity of the orbiting electron gives

$$m(v_4)^2 + \frac{Z\alpha\hbar\xi v_4}{r} = Bv_4 \,, \tag{5.41}$$

where $\alpha = \dfrac{e^2}{\hbar 4\pi\varepsilon_0 c}$ is the fine structure constant and where $\hbar = \dfrac{h}{2\pi}$ is the reduced Planck's constant. For circular orbits equation (5.34) is reduced to $mv^2 = Z\alpha\hbar\xi v_4/r$. Substitution of the latter in equation (5.41) gives $mc^2 = Bv_4$ or

$$Bc = m\gamma c^2 \,. \tag{5.42}$$

As mentioned earlier, Bc is equal to the total energy in the theory of relativity. In the AEST mass is a constant of motion, although it may change from one bound state to another. The AEST conservation laws will be applied to the transition of a state to another state by means of the absorption or emission of a massive photon. The initial state of the electron will be given an index n, the final state an index N. The conservation of mass for the emission of a photon reads

$$m_n - m_\gamma = m_N \,, \tag{5.43}$$

where m_γ is the mass of the photon. A plus sign would correspond to the absorption of a photon. The bound states in hydrogenic atoms are caused by the electric field of the nucleus. As the proper time momentum is conserved for free states, it is the proper time viriality B which is conserved for transitions between bound states. For a transition between bound states of the hydrogenic atom this is

$$B_n = B_N \,. \tag{5.44}$$

For state n and N we have

$$B_n = m_n\gamma_n c \qquad , \qquad B_N = m_N\gamma_N c \,. \tag{5.45}$$

Substitution in equation (5.44) leads to

$$\frac{m_n}{\sqrt{1-\frac{v_n^2}{c^2}}} = \frac{m_N}{\sqrt{1-\frac{v_N^2}{c^2}}} \,. \tag{5.46}$$

Again, it seems as if the mass of the orbiting electrons depend on their velocity in the same way as in the SRT. In an AEST, however, the change of mass is due to the emission or absorption of the mass of a photon. In case of elliptic orbits the mass of the electron is a constant of motion in the AEST. The mass of the electron will not change during an elliptical orbit despite the variation of the velocity.

For circular orbits the square of equation (5.34) reads

$$m_n^2 v_n^4 r_n^2 = Z^2 \alpha^2 \hbar^2 \xi^2 \left(c^2 - v_n^2\right) . \tag{5.47}$$

Taking the square of equation (5.35) and applying the quantisation condition $L = n\hbar$, we obtain

$$m_n^2 v_n^2 r_n^2 = n^2 \hbar^2 . \tag{5.48}$$

Eliminating mr from the latter two equations, we obtain

$$v_n^2 = \frac{Z^2 \alpha^2 \xi^2}{n^2 + Z^2 \alpha^2 \xi^2} c^2 \tag{5.49}$$

or

$$\frac{1}{1 - \frac{v_n^2}{c^2}} = 1 + \frac{Z^2 \alpha^2 \xi^2}{n^2} . \tag{5.50}$$

Similar expressions hold for state N. From equation (5.45) it follows

$$B_n = m_n c \sqrt{1 + \frac{Z^2 \alpha^2 \xi^2}{n^2}} \quad , \quad B_N = m_N c \sqrt{1 + \frac{Z^2 \alpha^2 \xi^2}{N^2}} . \tag{5.51}$$

From equation (5.44) it then follows

$$\frac{m_n^2}{m_N^2} = \frac{1 + \frac{Z^2 \alpha^2 \xi^2}{n^2}}{1 + \frac{Z^2 \alpha^2 \xi^2}{N^2}} . \tag{5.52}$$

The mass of a free electron, $n = \infty$, will be denoted as m_e Its value is the same as what is called the rest mass in the SRT. With $m_\infty = m_e$ it follows that the mass of the electron in state n and N are given by

$$m_n = \frac{m_e}{\sqrt{1 + \frac{Z^2 \alpha^2 \xi^2}{n^2}}} \quad , \quad m_N = \frac{m_e}{\sqrt{1 + \frac{Z^2 \alpha^2 \xi^2}{N^2}}} . \tag{5.53}$$

Furthermore, we find

$$B_n = B_N = m_e c . \tag{5.54}$$

For the radius of state n we obtain

$$r_n = \frac{n\hbar}{m_e} \frac{1}{\sqrt{1 - \frac{v_n^2}{c^2}}} = \frac{n\hbar}{m_e} \sqrt{1 + \frac{Z^2 \alpha^2 \xi^2}{n^2}} . \tag{5.55}$$

It follows that $v_N > v_n$ and $r_N < r_n$ if $N < n$. In particular $m_N < m_n$ if $N < n$, in correspondence with the classical picture of the emission of a photon when an electron makes a transition to a lower state. Substituting the expressions (5.53 for the electron masses into equation (5.43) and using the frequency-momentum relation for the photon, $m_\gamma c^2 = hf$, we obtain for the frequency of the emitted photon

$$f = \frac{m_e c^2}{h}\left(\frac{1}{\sqrt{1+\frac{Z^2\alpha^2\xi^2}{n^2}}} - \frac{1}{\sqrt{1+\frac{Z^2\alpha^2\xi^2}{n^2}}}\right) . \tag{5.56}$$

Neglecting terms of order $Z^4\alpha^4\xi^4/n^4$ it is reduced to

$$f = \frac{m_e c^2 Z^2\alpha^2\xi^2}{2h}\left(\frac{1}{N^2} - \frac{1}{n^2}\right) . \tag{5.57}$$

If $\xi \approx 1$, the latter is in agreement with Bohr's classical result.

5.5 Maxwell equations

For the derivation of the Maxwell equations we substitute equation (5.18) into equation (5.19) and equation (5.20) [31] . The result is

$$E_i = cF_{i4} = c\partial_i A_4 - c\partial_4 A_i \tag{5.58}$$

and

$$B_k = \frac{1}{2}\varepsilon_{ijk}F_{ij} = \varepsilon_{ijk}\partial_i A_j \tag{5.59}$$

respectively. From the latter two equations we obtain the AEST version of the homogeneous Maxwell equations

$$\partial_k B_k = \varepsilon_{ijk}\partial_k\partial_i A_j = 0\,, \tag{5.60}$$

$$\varepsilon_{ijk}\partial_j E_k = c\,\varepsilon_{ijk}\partial_j\partial_k A_4 - \partial_t\varepsilon_{ijk}\partial_j A_k = -\partial_t B_i\,. \tag{5.61}$$

In vector notation they read

$$\nabla \cdot \vec{B} = 0 \tag{5.62}$$

and

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0\,. \tag{5.63}$$

For the derivation of the other two Maxwell equations we consider the situation where the electromagnetic field is caused by a four current $j_\mu = \rho u_\mu$, where u_μ is the four velocity of the moving charge density which generates the electromagnetic field. That is

$$\partial_\nu F_{\mu\nu} = \mu_0 j_\mu\,, \tag{5.64}$$

where μ_0 is the vacuum permeability, $\varepsilon_0\mu_0 = c^{-2}$. Substituting the definitions (5.19) and (5.20), we obtain

$$\partial_\nu F_{4\nu} = \partial_k F_{4k} = -\frac{1}{c}\partial_k E_k = \mu_0 j_4 = \mu_0 \rho u_4 = \frac{sc\mu_0\rho}{\gamma(u)} \tag{5.65}$$

and

$$\partial_\nu F_{k\nu} = \partial_4 F_{k4} + \partial_i F_{ki} = \frac{1}{c}\partial_4 E_k + \varepsilon_{kim}\partial_i B_m = \mu_0 j_k\,, \tag{5.66}$$

where s is the sign of u_4 and where $1/\gamma(u) = \sqrt{1-u^2/c^2}$ with u the spatial velocity of the source current. The equations above lead to

$$\partial_i E_i = -\frac{s\rho}{\varepsilon_0\gamma(u)} \tag{5.67}$$

and

$$\varepsilon_{kim}\partial_i B_m + \frac{1}{c^2}\partial_t E_k = \mu_0 j_k\,. \tag{5.68}$$

In vector notation the latter two equations read

$$\nabla\cdot\vec{E} = -\frac{s\rho}{\varepsilon_0\gamma(u)} \tag{5.69}$$

and

$$\nabla\times\vec{B} + \frac{1}{c^2}\partial_t\vec{E} = \mu_0\vec{j}\,. \tag{5.70}$$

They are the AEST analogue of the inhomogeneous Maxwell equations.

Suppose $s = 1$ and $\gamma(u) \approx 1$. Then the time derivative of the second last equation and the ∇ of the last equation read

$$\frac{1}{c^2}\partial_t\nabla\cdot\vec{E} = -\mu_0\partial_t\rho \tag{5.71}$$

and

$$\nabla\cdot\big(\nabla\times\vec{B}\big) + \frac{1}{c^2}\partial_t\nabla\cdot\vec{E} = \mu_0\nabla\cdot\vec{j}\,. \tag{5.72}$$

Since $\nabla\cdot\big(\nabla\times\vec{B}\big) = 0$ the comparison of the layyer two equations gives

$$\partial_t\rho + \nabla\cdot\vec{j} = 0\,. \tag{5.73}$$

The latter is the continuity equation as desired. Although things seem consistent, there clearly is an issue with some signs in the AEST analogue of the inhomogeneous Maxwell equations.

Chapter 6

Geometric Algebra

6.1 Introduction

Geometric Algebra (Clifford algebra) is of use for physics and geometry [32] . It can be applied for the description of physics in a relative Minkowki spacetime. It can also been applied for the description of physics in a Euclidean spacetime [33, 34] . Before we apply it to the electrodynamics in an AEST, we first give a brief introduction to geometric algebra. We start with geometric algebra in the three spatial dimensions x, y and z.

The geometric algebra of three dimensions is generated by the frame of orthonormal basis vectors e_1, e_2, e_3: $\mathrm{e}_m\mathrm{e}_m = 1$ and $\mathrm{e}_m\mathrm{e}_n = -\mathrm{e}_n\mathrm{e}_m$. For the three dimensional frame the geometric algebra is spanned by a 8-dimensional basis: 1 scalar $\{1\}$, 3 vectors $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$, 3 bivectors $\{\mathrm{e}_1\mathrm{e}_2, \mathrm{e}_1\mathrm{e}_3, \mathrm{e}_2\mathrm{e}_3\}$ and 1 trivector $\{\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\}$. For the square of the bivectors we have $(\mathrm{e}_m\mathrm{e}_n)^2 = -1$, $n \neq m$. For the square of the trivector we have $(\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3)^2 = -1$. The geometric algebra is graded: a scalar is grade 0, a vector is grade 1, a bivector is grade 2, a trivector is grade 3. and so on. For a three dimensional frame the trivector is also called a pseudoscalar and we will denote it as I_3. The index is its dimension and allows to distinguish between pseudoscalars of different dimension: $I_1 = \mathrm{e}_1$ and $I_2 = \mathrm{e}_1\mathrm{e}_2$, $I_3 = \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3$, and so on. For the square of the pseudoscalars there holds $I_1^2 = 1$, $I_2^2 = -1$ and $I_3^2 = -1$. Since I_2 and I_3 square to -1, the bivector I_2 and trivector I_3 behave as the imaginary unit i or as $-i$.

The algebra of the three basis vectors can be summarised as

$$\mathrm{e}_m\mathrm{e}_n = \mathrm{e}_m \cdot \mathrm{e}_n + \mathrm{e}_m \wedge \mathrm{e}_n = \delta_{mn} + I_3\varepsilon_{mnk}\mathrm{e}_k \quad , \qquad m, n, k \in \{1, 2, 3\} , \tag{6.1}$$

where ε_{mnk} is the Levi-Civita symbol:

$$\varepsilon_{mnk} = \begin{cases} +1 & \text{if } (m, n, k) \text{ is an even permutation of } (1, 2, 3) , \\ -1 & \text{if } (m, n, k) \text{ is an odd permutation of } (1, 2, 3) , \\ 0 & \text{if } (m, n, k) \text{ is not a permutation of } (1, 2, 3) . \end{cases} \tag{6.2}$$

The algebra of the three basis vectors reminds us at the Pauli algebra of quantum mechanics. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{6.3}$$

The Pauli matrices satisfy the following algebra

$$\sigma_m \sigma_n = \delta_{mn} \sigma_0 + i \varepsilon_{mnk} \sigma_k \,, \tag{6.4}$$

with

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.5}$$

the 2×2 identity matrix. Indeed if I_3 is interpreted as the imaginary unit i, the algebra for the three basis vectors is identical to the Pauli algebra.

An arbitrary elements A of the 8-dimensional geometric algebra can be written as

$$A = a_0 + a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3 + a_{12} \mathrm{e}_1 \wedge \mathrm{e}_2 + a_{31} \mathrm{e}_3 \wedge \mathrm{e}_1 + a_{23} \mathrm{e}_2 \wedge \mathrm{e}_3 + a_{123} \mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3 \,. \tag{6.6}$$

It is equal to

$$A = a_0 + a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3 + a_{12} \mathrm{e}_1 \mathrm{e}_2 + a_{31} \mathrm{e}_3 \mathrm{e}_1 + a_{23} \mathrm{e}_2 \mathrm{e}_3 + a_{123} \mathrm{e}_1 \mathrm{e}_2 \mathrm{e}_3 \,. \tag{6.7}$$

We will adopt the notation of Pavšič by calling the element A a *polyvector* [35] . The coefficients are just numbers (scalars) and commute with all basis vectors. By means of the pseudoscalar I_3 the polyvector A can also be written as

$$A = a_0 + a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3 - I_3 a_{12} I_3 \mathrm{e}_1 \mathrm{e}_2 - I_3 a_{31} I_3 \mathrm{e}_3 \mathrm{e}_1 - I_3 a_{23} I_3 \mathrm{e}_2 \mathrm{e}_3 + I_3 a_{123} \,. \tag{6.8}$$

Substituting $\mathrm{e}_1 \mathrm{e}_2 \mathrm{e}_3$ for the right I_3 of each I_3 pair and performing the algebra, we obtain

$$A = a_0 + a_1 \mathrm{e}_1 + a_2 \mathrm{e}_2 + a_3 \mathrm{e}_3 + I_3 a_{12} \mathrm{e}_3 + I_3 a_{31} \mathrm{e}_2 + I_3 a_{23} \mathrm{e}_1 + I_3 a_{123} \,. \tag{6.9}$$

Substitution of i for I_3 gives

$$A = (a_0 + i a_{123}) + (a_1 + i a_{23}) \mathrm{e}_1 + (a_2 + i a_{31}) \mathrm{e}_2 + (a_3 + i a_{12}) \mathrm{e}_3 \,. \tag{6.10}$$

That is, the 8 dimensional basis can be represented by a four dimensional complex basis with one complex scalar and three vectors with complex coefficients.

The geometric product of two polyvectors A and B is

$$AB = A \cdot B + A \wedge B \,. \tag{6.11}$$

As a consequence,

$$A \cdot B = \frac{1}{2}\,(AB + BA)\ , \quad A \wedge B = \frac{1}{2}\,(AB - BA)\ . \tag{6.12}$$

Let a polyvector A be as given by equation (6.7) and let a polyvector B be given by

$$B = b_0 + b_1\mathrm{e}_1 + b_2\mathrm{e}_2 + b_3\mathrm{e}_3 + b_{12}\mathrm{e}_1\mathrm{e}_2 + b_{31}\mathrm{e}_3\mathrm{e}_1 + b_{23}\mathrm{e}_2\mathrm{e}_3 + b_{123}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\,. \tag{6.13}$$

The geometric product AB is a polyvector given by

$$\begin{aligned} AB \;=\; & (a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 - a_{12}b_{12} - a_{23}b_{23} - a_{31}b_{31} - a_{123}b_{123}) + \\ & (a_0b_1 + a_1b_0 - a_2b_{12} + a_{12}b_2 + a_3b_{31} - a_{31}b_3 - a_{23}b_{123} - a_{123}b_{23})\,\mathrm{e}_1 + \\ & (a_0b_2 + a_2b_0 - a_3b_{23} + a_{23}b_3 + a_1b_{12} - a_{12}b_1 - a_{31}b_{123} - a_{123}b_{31})\,\mathrm{e}_2 + \\ & (a_0b_3 + a_3b_0 - a_1b_{31} + a_{31}b_1 + a_2b_{23} - a_{23}b_2 - a_{12}b_{123} - a_{123}b_{12})\,\mathrm{e}_3 + \\ & (a_0b_{12} + a_{12}b_0 + a_1b_2 - a_2b_1 - a_{23}b_{31} + a_{31}b_{23} + a_3b_{123} + a_{123}b_3)\,\mathrm{e}_1\mathrm{e}_2 + \\ & (a_0b_{23} + a_{23}b_0 + a_2b_3 - a_3b_2 - a_{31}b_{12} + a_{12}b_{31} + a_1b_{123} + a_{123}b_1)\,\mathrm{e}_2\mathrm{e}_3 + \\ & (a_0b_{31} + a_{31}b_0 + a_3b_1 - a_1b_3 - a_{12}b_{23} + a_{23}b_{12} + a_2b_{123} + a_{123}b_2)\,\mathrm{e}_3\mathrm{e}_1 + \\ & (a_0b_{123} + a_{123}b_0 + a_1b_{23} + a_{23}b_1 + a_2b_{31} + a_{31}b_2 + a_3b_{12} + a_{12}b_3)\,\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\,. \end{aligned} \tag{6.14}$$

For the inner product and the wedge product there holds

$$\begin{aligned} A \cdot B \;=\; & (a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 - a_{12}b_{12} - a_{23}b_{23} - a_{31}b_{31} - a_{123}b_{123}) + \\ & (a_0b_1 + a_1b_0 - a_{23}b_{123} - a_{123}b_{23})\,\mathrm{e}_1 + (a_0b_2 + a_2b_0 - a_{31}b_{123} - a_{123}b_{31})\,\mathrm{e}_2 + \\ & (a_0b_3 + a_3b_0 - a_{12}b_{123} - a_{123}b_{12})\,\mathrm{e}_3 + (a_0b_{12} + a_{12}b_0 + a_3b_{123} + a_{123}b_3)\,\mathrm{e}_1\mathrm{e}_2 + \\ & (a_0b_{23} + a_{23}b_0 + a_1b_{123} + a_{123}b_1)\,\mathrm{e}_2\mathrm{e}_3 + (a_0b_{31} + a_{31}b_0 + a_2b_{123} + a_{123}b_2)\,\mathrm{e}_3\mathrm{e}_1 + \\ & (a_0b_{123} + a_{123}b_0 + a_1b_{23} + a_{23}b_1 + a_2b_{31} + a_{31}b_2 + a_3b_{12} + a_{12}b_3)\,\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} A \wedge B \;=\; & (-a_2b_{12} + a_{12}b_2 + a_3b_{31} - a_{31}b_3)\,\mathrm{e}_1 + (-a_3b_{23} + a_{23}b_3 + a_1b_{12} - a_{12}b_1)\,\mathrm{e}_2 + \\ & (-a_1b_{31} + a_{31}b_1 + a_2b_{23} - a_{23}b_2)\,\mathrm{e}_3 + (a_1b_2 - a_2b_1 - a_{23}b_{31} + a_{31}b_{23})\,\mathrm{e}_1\mathrm{e}_2 + \\ & (a_2b_3 - a_3b_2 - a_{31}b_{12} + a_{12}b_{31})\,\mathrm{e}_2\mathrm{e}_3 + \\ & (a_3b_1 - a_1b_3 - a_{12}b_{23} + a_{23}b_{12})\,\mathrm{e}_3\mathrm{e}_1 \end{aligned} \tag{6.16}$$

respectively.

If, for instance, $v = v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3$ is a vector and $B = b_1\mathrm{e}_2\mathrm{e}_3 + b_2\mathrm{e}_3\mathrm{e}_1 + b_3\mathrm{e}_1\mathrm{e}_2$ is a bivector, then

$$v \cdot B = (v_1b_1 + v_2b_2 + v_3b_3)\,\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\,. \tag{6.17}$$

and

$$v \wedge B = (v_3b_2 - v_2b_3)\,\mathrm{e}_1 + (v_1b_3 - v_3b_1)\,\mathrm{e}_2 + (v_2b_1 - v_1b_2)\,\mathrm{e}_3 \tag{6.18}$$

In vector notation

$$\vec{v} \cdot B = I_3 \left(\vec{v} \cdot \vec{b}\right) , \quad \vec{v} \wedge B = -\vec{v} \times \vec{b} . \tag{6.19}$$

The geometric algebra of the four dimensions, x, y, z and $c\tau$ of the AEST, is generated by the frame of orthonormal vectors e_1, e_2, e_3, e_4: $\mathrm{e}_\mu \mathrm{e}_\mu = 1$ and $\mathrm{e}_\mu \mathrm{e}_\nu = -\mathrm{e}_\nu \mathrm{e}_\mu$. For the four dimensional frame the geometric algebra is spanned by a 16-dimensional basis: 1 scalar $\{1\}$, 4 vectors $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\}$, 6 bivectors $\{\mathrm{e}_1\mathrm{e}_2, \mathrm{e}_1\mathrm{e}_3, \mathrm{e}_1\mathrm{e}_4, \mathrm{e}_2\mathrm{e}_3, \mathrm{e}_2\mathrm{e}_4, \mathrm{e}_3\mathrm{e}_4\}$, 4 trivectors $\{\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3, \mathrm{e}_1\mathrm{e}_2\mathrm{e}_4, \mathrm{e}_1\mathrm{e}_3\mathrm{e}_4, \mathrm{e}_2\mathrm{e}_3\mathrm{e}_4, \}$ and 1 quadrivector $\{\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4\}$. The bivectors square to -1. The trivectors square to -1. The quadrivector is also called a pseudoscalar and we will denote it as I_4. For the square of the quadrivector we have $I_4^2 = (\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4)^2 = 1$.

For the 16-dimensional algebra a polyvector A is in general written as

$$\begin{aligned} A &= \alpha_0 + \alpha_1\mathrm{e}_1 + \alpha_2\mathrm{e}_2 + \alpha_3\mathrm{e}_3 + \alpha_4\mathrm{e}_4 + \alpha_{12}\mathrm{e}_1\mathrm{e}_2 + \alpha_{31}\mathrm{e}_3\mathrm{e}_1 + \alpha_{23}\mathrm{e}_2\mathrm{e}_3 + \alpha_{14}\mathrm{e}_1\mathrm{e}_4 + \alpha_{24}\mathrm{e}_2\mathrm{e}_4 + \\ &\quad + \alpha_{34}\mathrm{e}_3\mathrm{e}_4 + \alpha_{123}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + \alpha_{124}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + \alpha_{134}\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 + \alpha_{234}\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + \alpha_{1234}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 . \end{aligned} \tag{6.20}$$

Since I_4 squares to 1 we do not essentially change the polyvector A if we take the product of I_4^2 with some elements of A:

$$\begin{aligned} A &= \alpha_0 + \alpha_1\mathrm{e}_1 + \alpha_2\mathrm{e}_2 + \alpha_3\mathrm{e}_3 + \alpha_4\mathrm{e}_4 + \alpha_{12}\mathrm{e}_1\mathrm{e}_2 + \alpha_{31}\mathrm{e}_3\mathrm{e}_1 + \alpha_{23}\mathrm{e}_2\mathrm{e}_3 + \alpha_{14}I_4^2\mathrm{e}_1\mathrm{e}_4 + \\ &\quad \alpha_{24}I_4^2\mathrm{e}_2\mathrm{e}_4 + \alpha_{34}I_4^2\mathrm{e}_3\mathrm{e}_4 + \alpha_{123}I_4^2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + \alpha_{124}I_4^2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + \alpha_{134}I_4^2\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 + \\ &\quad \alpha_{234}I_4^2\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + \alpha_{1234}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 . \end{aligned} \tag{6.21}$$

According to the algebra we can write the element $\alpha_{123}I_4^2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3$ as $I_4\alpha_{123}\mathrm{e}_4$. Performing the algebra to all the other elements containing I_4^2 we obtain

$$\begin{aligned} A &= (\alpha_0 + I_4\alpha_{1234}) + (\alpha_1 - I_4\alpha_{234})\,\mathrm{e}_1 + (\alpha_2 + I_4\alpha_{134})\,\mathrm{e}_2 + (\alpha_3 - I_4\alpha_{124})\,\mathrm{e}_3 + \\ &\quad (\alpha_4 + I_4\alpha_{123})\,\mathrm{e}_4 + (\alpha_{12} - I_4\alpha_{34})\,\mathrm{e}_1\mathrm{e}_2 + (\alpha_{31} - I_4\alpha_{24})\,\mathrm{e}_3\mathrm{e}_1 + \\ &\quad (\alpha_{23} - I_4\alpha_{14})\,\mathrm{e}_2\mathrm{e}_3 . \end{aligned} \tag{6.22}$$

One can also multiply some elements of A with $-I_3^2$:

$$\begin{aligned} A &= \alpha_0 + \alpha_1\mathrm{e}_1 + \alpha_2\mathrm{e}_2 + \alpha_3\mathrm{e}_3 + \alpha_4\mathrm{e}_4 + \alpha_{12}\mathrm{e}_1\mathrm{e}_2 + \alpha_{31}\mathrm{e}_3\mathrm{e}_1 + \alpha_{23}\mathrm{e}_2\mathrm{e}_3 + \alpha_{14}\mathrm{e}_1\mathrm{e}_4 + \alpha_{24}\mathrm{e}_2\mathrm{e}_4 + \\ &\quad + \alpha_{34}\mathrm{e}_3\mathrm{e}_4 + \alpha_{123}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 - I_3^2\alpha_{124}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 - I_3^2\alpha_{134}\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 - I_3^2\alpha_{234}\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + \\ &\quad \alpha_{1234}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 . \end{aligned} \tag{6.23}$$

Then we obtain

$$\begin{aligned} A &= (\alpha_0 +_{123} I_3) + \alpha_1\mathrm{e}_1 + \alpha_2\mathrm{e}_2 + \alpha_3\mathrm{e}_3 + (\alpha_4 + I_3\alpha_{1234})\mathrm{e}_4 + \alpha_{12}\mathrm{e}_1\mathrm{e}_2 + \alpha_{31}\mathrm{e}_3\mathrm{e}_1 + \\ &\quad \alpha_{23}\mathrm{e}_2\mathrm{e}_3 + (\alpha_{14} + I_3\alpha_{234})\mathrm{e}_1\mathrm{e}_4 + (\alpha_{24} - I_3\alpha_{134})\mathrm{e}_2\mathrm{e}_4 + (\alpha_{34} + I_3\alpha_{124})\mathrm{e}_3\mathrm{e}_4 . \end{aligned} \tag{6.24}$$

If we Interpret I_3 as the imaginary unit i, then one can interpret A as a polyvector with a scalar, four vectors and six bivectors with complex coefficients.

6.2 Rotors

An important concept of geometric algebra is the description of rotations by means of a *rotor*. A rotation of a vector $v = v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4$ through an angle α in a plane B is given by

$$v' = RvR^\dagger = e^{-B\alpha/2} v e^{B\alpha/2}\,, \tag{6.25}$$

where $R = e^{-B\alpha/2}$ is the rotor. If we take, for instance, the bivector B equal to the $\mathrm{e}_1\mathrm{e}_2$ plane, then

$$v' = e^{-\mathrm{e}_1\mathrm{e}_2\alpha_{12}/2} v e^{\mathrm{e}_1\mathrm{e}_2\alpha_{12}/2} = e^{-I_2\alpha_{12}/2} v e^{I_2\alpha_{12}/2}\,. \tag{6.26}$$

Since the bivector $I_2 = \mathrm{e}_1\mathrm{e}_2$ behaves as the imaginary unit i we can write

$$v' = \Big(\cos(\alpha_{12}/2) - I_2\sin(\alpha_{12}/2)\Big) v \Big(\cos(\alpha_{12}/2) + I_2\sin(\alpha_{12}/2)\Big)\,. \tag{6.27}$$

For a rotation $\alpha_{12} = \pi$ in the $\mathrm{e}_1\mathrm{e}_2$ plane, we obtain

$$\begin{aligned} v' &= \Big(\cos(\pi/2) - I_2\sin(\pi/2)\Big)\,(v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4)\,\Big(\cos(\pi/2) + I_2\sin(\pi/2)\Big) \\ &= -\mathrm{e}_1\mathrm{e}_2\,(v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4)\,\mathrm{e}_1\mathrm{e}_2 \\ &= -v_1\mathrm{e}_1\mathrm{e}_2\mathrm{e}_1\mathrm{e}_1\mathrm{e}_2 - v_2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_2\mathrm{e}_1\mathrm{e}_2 - v_3\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_1\mathrm{e}_2 - v_4\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4\mathrm{e}_1\mathrm{e}_2 \\ &= -v_1\mathrm{e}_1 - v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4\,. \end{aligned} \tag{6.28}$$

So, the rotation generated by the rotor $e^{-\mathrm{e}_1\mathrm{e}_2\pi/2}$ has reversed the projection of v on the $\mathrm{e}_1\mathrm{e}_2$ plane. Indeed it corresponds with a rotation through an angle π in the $\mathrm{e}_1\mathrm{e}_2$ plane.

For the present purpose we are interested in infinitesimally small rotations. Let $v' = v + \Delta v$ be the result of the rotation of vector v through a small angle $\Delta\alpha_{12}$ in the $\mathrm{e}_1\mathrm{e}_2$ plane. That is,

$$v + \Delta v = e^{-\mathrm{e}_1\mathrm{e}_2\Delta\alpha_{12}/2} v e^{\mathrm{e}_1\mathrm{e}_2\Delta\alpha_{12}/2}\,. \tag{6.29}$$

That is

$$\begin{aligned} v + \Delta v &= \Big(\cos(\Delta\alpha_{12}/2) - \mathrm{e}_1\mathrm{e}_2\sin(\Delta\alpha_{12}/2)\Big) v \Big(\cos(\Delta\alpha_{12}/2) + \mathrm{e}_1\mathrm{e}_2\sin(\Delta\alpha_{12}/2)\Big) \\ &= v + \Big(v\mathrm{e}_1\mathrm{e}_2 - \mathrm{e}_1\mathrm{e}_2 v\Big)\sin(\Delta\alpha_{12}/2)\cos(\Delta\alpha_{12}/2)\,. \end{aligned} \tag{6.30}$$

To first order in $\Delta\alpha_{12}$ this is

$$\Delta v = \frac{1}{2}\Big(v\mathrm{e}_1\mathrm{e}_2 - \mathrm{e}_1\mathrm{e}_2 v\Big)\Delta\alpha_{12}\,. \tag{6.31}$$

Substituting $v = (v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4)$ we obtain

$$\Delta v = (v_1\mathrm{e}_2 - v_2\mathrm{e}_1)\,\Delta\alpha_{12}\,. \tag{6.32}$$

If the rotation took place during a small time lapse Δt we have

$$\frac{\Delta v}{\Delta t} = (v_1\mathrm{e}_2 - v_2\mathrm{e}_1)\,\frac{\Delta\alpha_{12}}{\Delta t}\,. \tag{6.33}$$

In the infinitesimal limit we obtain for the acceleration

$$a = \dot{v} = (v_1 e_2 - v_2 e_1)\,\omega_{12}\,. \tag{6.34}$$

where

$$\omega_{12} = \dot{\alpha}_{12}\,. \tag{6.35}$$

We can take all the other rotations into consideration in a similar way as in the first section of this chapter. The result is

$$a_\rho e_\rho = v_\sigma \omega_{\sigma\rho} e_\rho = v_\sigma e_\sigma \omega_{\sigma\rho} e_\sigma e_\rho\,. \tag{6.36}$$

Explicitly

$$\begin{aligned} a \;=\; & (v_2\omega_{21} + v_3\omega_{31} + v_4\omega_{41})\, e_1 + (v_1\omega_{12} + v_3\omega_{32} + v_4\omega_{42})\, e_2 + \\ & (v_1\omega_{13} + v_2\omega_{23} + v_4\omega_{43})\, e_3 + (v_1\omega_{14} + v_2\omega_{24} + v_3\omega_{34})\, e_4\,. \end{aligned} \tag{6.37}$$

As before, we let the $\omega_{\mu\nu}$ be caused by fields $F_{\nu\mu}$. That is, for $\omega_{\mu\nu} e_\mu e_\nu = \frac{q}{m} F_{\nu\mu} e_\mu e_\nu$ we obtain for the force

$$K_\mu e_\mu = m a_\mu e_\mu = q v_\nu F_{\mu\nu} e_\mu\,. \tag{6.38}$$

6.3 Lorentz force with geometric algebra

The electromagnetic field F is grade 2 and antisymmetric: $F = f_{\mu\nu} e_\mu e_\nu$ with $f_{\nu\mu} = -f_{\mu\nu}$. Explicitly,

$$F = f_{4j} e_4 e_j + f_{j4} e_j e_4 + f_{12} e_1 e_2 + f_{21} e_2 e_1 + f_{23} e_2 e_3 + f_{32} e_3 e_2 + f_{31} e_3 e_1 + f_{13} e_1 e_3\,, \tag{6.39}$$

which is equal to

$$F = 2f_{14} e_1 e_4 + 2f_{24} e_2 e_4 + 2f_{34} e_3 e_4 + 2f_{12} e_1 e_2 + 2f_{23} e_2 e_3 + 2f_{31} e_3 e_1\,. \tag{6.40}$$

Now let

$$2f_{j4} = \frac{E_j}{c} \tag{6.41}$$

and

$$2f_{ij} = \varepsilon_{ijk} B_k\,. \tag{6.42}$$

Then

$$F = \frac{E_1}{c} e_1 e_4 + \frac{E_2}{c} e_2 e_4 + \frac{E_3}{c} e_3 e_4 + B_1 e_2 e_3 + B_2 e_3 e_1 + B_3 e_1 e_2\,. \tag{6.43}$$

First we consider the electromagnetic force K as given by

$$K = q\, F \wedge v\,. \tag{6.44}$$

Explicitly

$$\begin{aligned}\frac{1}{q}K_\mu \mathrm{e}_\mu &= \Big(\frac{E_1}{c}\mathrm{e}_1\mathrm{e}_4 + \frac{E_2}{c}\mathrm{e}_2\mathrm{e}_4 + \frac{E_3}{c}\mathrm{e}_3\mathrm{e}_4 + B_1\mathrm{e}_2\mathrm{e}_3 + B_2\mathrm{e}_3\mathrm{e}_1 + B_3\mathrm{e}_1\mathrm{e}_2\Big) \wedge \\ &\Big(v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4\Big) = -v_1\frac{E_1}{c}\mathrm{e}_4 - v_2\frac{E_2}{c}\mathrm{e}_4 - v_3\frac{E_3}{c}\mathrm{e}_4 + \\ &v_4\frac{E_1}{c}\mathrm{e}_1 + v_4\frac{E_2}{c}\mathrm{e}_2 + v_4\frac{E_3}{c}\mathrm{e}_3 + v_1B_2\mathrm{e}_3 - v_1B_3\mathrm{e}_2 - v_2B_1\mathrm{e}_3 + \\ &v_2B_3\mathrm{e}_1 + v_3B_1\mathrm{e}_2 - v_3B_2\mathrm{e}_1\,. \end{aligned} \tag{6.45}$$

Grouping the coefficients for each basis vector leads to

$$\begin{aligned}\frac{1}{q}K_\mu \mathrm{e}_\mu &= \left(v_4\frac{E_1}{c} + v_2B_3 - v_3B_2\right)\mathrm{e}_1 + \left(v_4\frac{E_2}{c} - v_1B_3 + v_3B_1\right)\mathrm{e}_2 + \\ &\left(v_4\frac{E_3}{c} + v_1B_2 - v_2B_1\right)\mathrm{e}_3 - \left(v_1\frac{E_1}{c} + v_2\frac{E_2}{c} + v_3\frac{E_3}{c}\right)\mathrm{e}_4\,. \end{aligned} \tag{6.46}$$

That is,

$$K_4 = -q\,v_j\frac{E_j}{c}\,, \tag{6.47}$$

$$K_i = q\left(v_4\frac{E_i}{c} + \varepsilon_{ijk}v_jB_k\right)\,. \tag{6.48}$$

It is the AEST analogue of the Lorentz force. In vector notation it reads

$$\vec{K} = q\left(v_4\frac{\vec{E}}{c} + \vec{v}\times\vec{B}\right)\,. \tag{6.49}$$

Let M be the mysterious force given by:

$$M = q\,F\cdot v\,. \tag{6.50}$$

For M we obtain

$$\begin{aligned}\frac{1}{q}M_\mu \mathrm{e}_\mu &= \Big(\frac{E_1}{c}\mathrm{e}_1\mathrm{e}_4 + \frac{E_2}{c}\mathrm{e}_2\mathrm{e}_4 + \frac{E_3}{c}\mathrm{e}_3\mathrm{e}_4 + B_1\mathrm{e}_2\mathrm{e}_3 + B_2\mathrm{e}_3\mathrm{e}_1 + B_3\mathrm{e}_1\mathrm{e}_2\Big) \cdot \\ &\Big(v_1\mathrm{e}_1 + v_2\mathrm{e}_2 + v_3\mathrm{e}_3 + v_4\mathrm{e}_4\Big) = v_1\frac{E_2}{c}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + v_1\frac{E_3}{c}\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 + v_1B_1\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + \\ &-v_2\frac{E_1}{c}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + v_2\frac{E_3}{c}\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + v_2B_2\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 - v_3\frac{E_1}{c}\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 - v_3\frac{E_2}{c}\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + \\ &v_3B_3\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + v_4B_3\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + v_4B_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 - v_4B_2\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4\,. \end{aligned} \tag{6.51}$$

By means of $1 = I_4^2 = I_4\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4$ it can be written as

$$\begin{aligned}\frac{1}{q}M_\mu \mathrm{e}_\mu &= I_4\Big(- v_1\frac{E_2}{c}\mathrm{e}_3 + v_1\frac{E_3}{c}\mathrm{e}_2 + v_1B_1\mathrm{e}_4 + v_2\frac{E_1}{c}\mathrm{e}_3 - v_2\frac{E_3}{c}\mathrm{e}_1 + v_2B_2\mathrm{e}_4 + \\ &-v_3\frac{E_1}{c}\mathrm{e}_2 + v_3\frac{E_2}{c}\mathrm{e}_1 + v_3B_3\mathrm{e}_4 - v_4B_1\mathrm{e}_1 - v_4B_2\mathrm{e}_2 - v_4B_3\mathrm{e}_3\Big)\,. \end{aligned} \tag{6.52}$$

Grouping the coefficients for basis vector leads to

$$\begin{aligned}\frac{1}{q}M_\mu \mathrm{e}_\mu &= -I_4\left(v_2\frac{E_3}{c}-v_3\frac{E_2}{c}+v_4B_1\right)\mathrm{e}_1 - I_4\left(v_3\frac{E_1}{c}-v_1\frac{E_3}{c}+v_4B_2\right)\mathrm{e}_2 + \\ &\quad -I_4\left(v_1\frac{E_2}{c}-v_2\frac{E_1}{c}+v_4B_3\right)\mathrm{e}_3 + I_4\Big(v_1B_1+v_2B_2+v_3B_3\Big)\mathrm{e}_4\,. \end{aligned} \tag{6.53}$$

That is,

$$M_4 = I_4\, q\, v_j B_j\,, \tag{6.54}$$

$$M_k = -I_4\, q\left(v_4 B_k + \varepsilon_{kmn} v_m \frac{E_n}{c}\right)\,. \tag{6.55}$$

In vector notation the latter two equations read:

$$M_4 = I_4\, q\, \vec{v}\cdot\vec{B} \tag{6.56}$$

and

$$\vec{M} = -I_4\, q\left(v_4\vec{B} + \vec{v}\times\frac{\vec{E}}{c}\right)\,. \tag{6.57}$$

For the geometric product $Fv = F\cdot v + F\wedge v = (M+K)/q$ we obtain

$$Fv = v_4\vec{\mathcal{F}} - \vec{\mathcal{F}}\cdot\vec{v} + I_4\,\vec{\mathcal{F}}\times\vec{v}\,, \tag{6.58}$$

where

$$\mathcal{F} = F\mathrm{e}_4 = \left(\frac{E_k}{c} - I_4 B_k\right)\mathrm{e}_k\,. \tag{6.59}$$

6.4 Alternative derivation of the Lorentz force

An alternative derivation of the Lorentz force is achieved by changing to a bivector basis $\boldsymbol{\sigma}_k = \mathrm{e}_k\mathrm{e}_4$. To this end we multiply the vector v by e_4. The result is

$$v\mathrm{e}_4 = v_1\mathrm{e}_1\mathrm{e}_4 + v_2\mathrm{e}_2\mathrm{e}_4 + v_3\mathrm{e}_3\mathrm{e}_4 + v_4 = v_1\boldsymbol{\sigma}_1 + v_2\boldsymbol{\sigma}_2 + v_3\boldsymbol{\sigma}_3 + v_4 = v_m\boldsymbol{\sigma}_m m + v_4\,, \tag{6.60}$$

The $\boldsymbol{\sigma}_k$ are written in boldface to distinguish them from the Pauli matrices. For the algebra of the $\boldsymbol{\sigma}_k$ it follows that $\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_k = -1$ and $\boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j = I_4\varepsilon_{ijk}\boldsymbol{\sigma}_k$. It can be summarised as

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = -\delta_{ij} + I_4\varepsilon_{ijk}\boldsymbol{\sigma}_k\,. \tag{6.61}$$

By means of the $\boldsymbol{\sigma}_k$ and I_4 we can write the electromagnetic field as

$$F = \mathcal{F}\mathrm{e}_4 = \left(\frac{E_k}{c} - I_4B_k\right)\mathrm{e}_k\mathrm{e}_4 = \left(\frac{E_k}{c} - I_4B_k\right)\boldsymbol{\sigma}_k\,. \tag{6.62}$$

For $F\wedge(v\mathrm{e}_4)$ we have

$$F\wedge(v\mathrm{e}_4) = \left(\frac{E_k}{c} - I_4B_k\right)\boldsymbol{\sigma}_k\wedge\Big(v_m\boldsymbol{\sigma}_m + v_4\Big) = I_4\varepsilon_{kmn}\left(\frac{E_k}{c} - I_4B_k\right)v_m\boldsymbol{\sigma}_n\,. \tag{6.63}$$

For $F \cdot (v\mathrm{e}_4)$ we have

$$\begin{aligned} F \cdot (v\mathrm{e}_4) &= \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k \cdot \left(v_m \boldsymbol{\sigma}_m + v_4\right) \\ &= v_4 \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k - v_k \left(\frac{E_k}{c} - I_4 B_k\right) . \end{aligned} \tag{6.64}$$

As a consequence

$$F v\mathrm{e}_4 = v_4 \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k - v_k \left(\frac{E_k}{c} - I_4 B_k\right) + I_4 \varepsilon_{kmn} \left(\frac{E_k}{c} - I_4 B_k\right) v_m \boldsymbol{\sigma}_n . \tag{6.65}$$

It can be split in a scalar part

$$- v_k \frac{E_k}{c} = \frac{1}{q} K_4 , \tag{6.66}$$

a bivector part in the $\boldsymbol{\sigma}_k$ direction

$$v_4 \frac{E_k}{c} \boldsymbol{\sigma}_k + \varepsilon_{ijk} v_i B_j \boldsymbol{\sigma}_k = \frac{1}{q} K_k \boldsymbol{\sigma}_k , \tag{6.67}$$

a bivector part in the $I_4 \boldsymbol{\sigma}_k$ direction

$$- I_4 v_4 B_k \boldsymbol{\sigma}_k - I_4 \varepsilon_{ijk} v_i E_j \boldsymbol{\sigma}_k = \frac{1}{q} M_k \boldsymbol{\sigma}_k \tag{6.68}$$

and a quadrivector or pseudoscalar part

$$I_4 v_k B_k = \frac{1}{q} M_4 . \tag{6.69}$$

Obviously we end up with the same forces as before. So, the alternative approach does not deliver something new.

6.5 Maxwell equations with geometric algebra

We start considering $\partial \wedge F$ with F as defined in the previous section.

$$\begin{aligned} \partial \wedge F = {} & \left(\partial_1 \mathrm{e}_1 + \partial_2 \mathrm{e}_2 + \partial_3 \mathrm{e}_3 + \partial_4 \mathrm{e}_4\right) \wedge \left(\frac{E_1}{c} \mathrm{e}_1 \mathrm{e}_4 + \frac{E_2}{c} \mathrm{e}_2 \mathrm{e}_4 + \frac{E_3}{c} \mathrm{e}_3 \mathrm{e}_4 + B_1 \mathrm{e}_2 \mathrm{e}_3 + \right. \\ & \left. B_2 \mathrm{e}_3 \mathrm{e}_1 + B_3 \mathrm{e}_1 \mathrm{e}_2\right) = \partial_1 \frac{E_1}{c} \mathrm{e}_4 + \partial_2 \frac{E_2}{c} \mathrm{e}_4 + \partial_3 \frac{E_3}{c} \mathrm{e}_4 - \partial_4 \frac{E_1}{c} \mathrm{e}_1 + \\ & - \partial_4 \frac{E_2}{c} \mathrm{e}_2 - \partial_4 \frac{E_3}{c} \mathrm{e}_3 + \partial_1 B_3 \mathrm{e}_2 - \partial_1 B_2 \mathrm{e}_3 - \partial_2 B_3 \mathrm{e}_1 + \partial_2 B_1 \mathrm{e}_3 + \\ & - \partial_3 B_1 \mathrm{e}_2 + \partial_3 B_2 \mathrm{e}_1 . \end{aligned} \tag{6.70}$$

Grouping the coefficients for each basis vector leads to

$$\begin{aligned} \partial \wedge F = {} & - \left(\partial_4 \frac{E_1}{c} + \partial_2 B_3 - \partial_3 B_2\right) \mathrm{e}_1 - \left(\partial_4 \frac{E_2}{c} - \partial_1 B_3 + \partial_3 B_1\right) \mathrm{e}_2 + \\ & - \left(\partial_4 \frac{E_3}{c} + \partial_1 B_2 - \partial_2 B_1\right) \mathrm{e}_3 + \left(\partial_1 \frac{E_1}{c} + \partial_2 \frac{E_2}{c} + \partial_3 \frac{E_3}{c}\right) \mathrm{e}_4 . \end{aligned} \tag{6.71}$$

That is,

$$\partial \wedge F = -\varepsilon_{ijk}\partial_j B_k \mathrm{e}_i - \partial_4 \frac{E_i}{c}\mathrm{e}_i + \partial_j \frac{E_j}{c}\mathrm{e}_4 \,. \tag{6.72}$$

The equation

$$\partial \wedge F = -\mu_0 J \tag{6.73}$$

leads to

$$\partial_j \frac{E_j}{c} = -\mu_0 J_4 \tag{6.74}$$

and

$$\varepsilon_{ijk}\partial_j B_k + \partial_4 \frac{E_i}{c} = \mu_0 J_i \,. \tag{6.75}$$

In vector notation they read

$$\nabla \cdot \vec{E} = -c\mu_0 J_4 = -\frac{s\rho}{\varepsilon_0 \gamma(u)} \,, \tag{6.76}$$

$$\nabla \times \vec{B} + \frac{1}{c^2}\partial_t \vec{E} = \mu_0 \vec{j} \,. \tag{6.77}$$

We see the result is identical to the one derived with the tensor approach in the previous chapter.

One can also consider $\partial \cdot F$:

$$\begin{aligned} \partial \cdot F \;=\; & \Big(\partial_1 \mathrm{e}_1 + \partial_2 \mathrm{e}_2 + \partial_3 \mathrm{e}_3 + \partial_4 \mathrm{e}_4\Big) \cdot \Big(\frac{E_1}{c}\mathrm{e}_1\mathrm{e}_4 + \frac{E_2}{c}\mathrm{e}_2\mathrm{e}_4 + \frac{E_3}{c}\mathrm{e}_3\mathrm{e}_4 + B_1 \mathrm{e}_2\mathrm{e}_3 + \\ & B_2 \mathrm{e}_3\mathrm{e}_1 + B_3 \mathrm{e}_1\mathrm{e}_2\Big) = \partial_1 \frac{E_2}{c}\mathrm{e}_1\mathrm{e}_2\mathrm{e}_4 + \partial_1 \frac{E_3}{c}\mathrm{e}_1\mathrm{e}_3\mathrm{e}_4 + \partial_1 B_1 \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + \\ & \partial_2 \frac{E_1}{c}\mathrm{e}_2\mathrm{e}_1\mathrm{e}_4 + \partial_2 \frac{E_3}{c}\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 + \partial_2 B_2 \mathrm{e}_2\mathrm{e}_3\mathrm{e}_1 + \partial_3 \frac{E_1}{c}\mathrm{e}_3\mathrm{e}_1\mathrm{e}_4 + \partial_3 \frac{E_2}{c}\mathrm{e}_3\mathrm{e}_2\mathrm{e}_4 + \\ & \partial_3 B_3 \mathrm{e}_3\mathrm{e}_1\mathrm{e}_2 + \partial_4 B_3 \mathrm{e}_4\mathrm{e}_1\mathrm{e}_2 + \partial_4 B_1 \mathrm{e}_4\mathrm{e}_2\mathrm{e}_3 + \partial_4 B_2 \mathrm{e}_4\mathrm{e}_3\mathrm{e}_1 \,. \end{aligned} \tag{6.78}$$

By means of $I_4^2 = I_4 \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4 = 1$ it can be written as

$$\begin{aligned} \partial \cdot F \;=\; & I_4\Big(-\partial_1 \frac{E_2}{c}\mathrm{e}_3 + \partial_1 \frac{E_3}{c}\mathrm{e}_2 + \partial_1 B_1 \mathrm{e}_4 + \partial_2 \frac{E_1}{c}\mathrm{e}_3 - \partial_2 \frac{E_3}{c}\mathrm{e}_1 + \partial_2 B_2 \mathrm{e}_4 + \\ & -\partial_3 \frac{E_1}{c}\mathrm{e}_2 + \partial_3 \frac{E_2}{c}\mathrm{e}_1 + \partial_3 B_3 \mathrm{e}_4 - \partial_4 B_3 \mathrm{e}_3 - \partial_4 B_1 \mathrm{e}_1 - \partial_4 B_2 \mathrm{e}_2\Big) \,. \end{aligned} \tag{6.79}$$

Grouping the coefficients for basis vector leads to

$$\begin{aligned} \partial \cdot F \;=-\; & I_4 \left(\partial_2 \frac{E_3}{c} - \partial_3 \frac{E_2}{c} + \partial_4 B_1\right)\mathrm{e}_1 - I_4 \left(\partial_3 \frac{E_1}{c} - \partial_1 \frac{E_3}{c} + \partial_4 B_2\right)\mathrm{e}_2 + \\ & -I_4 \left(\partial_1 \frac{E_2}{c} - \partial_2 \frac{E_1}{c} + \partial_4 B_3\right)\mathrm{e}_3 + I_4\Big(\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3\Big)\mathrm{e}_4 \,. \end{aligned} \tag{6.80}$$

The equation $\partial \cdot F = 0$ leads to

$$\partial_j B_j = 0 \tag{6.81}$$

and

$$\partial_4 B_k + \varepsilon_{kmn}\partial_m \frac{E_n}{c} = 0\,. \tag{6.82}$$

In vector notation the latter two equations read:

$$\nabla \cdot \vec{B} = 0 \tag{6.83}$$

and

$$\partial_t \vec{B} + \nabla \times \vec{E} = 0\,. \tag{6.84}$$

They are the homogeneous Maxwell equations. Also here the result is identical to the one derived with the tensor approach in the previous chapter.

For the geometric product $\partial F = \partial \cdot F + \partial \wedge F$ we obtain

$$\partial F = \nabla \cdot \vec{\mathcal{F}}^\dagger - \partial_4 \vec{\mathcal{F}}^\dagger - I_4\, \nabla \times \vec{\mathcal{F}}^\dagger\,, \tag{6.85}$$

where $\mathcal{F}^\dagger = \mathrm{e}_4 F = -\vec{E}/c - I_4 \vec{B}$.

The equations $\partial \cdot F = 0$ and $\partial \wedge F = \mu_0 J$ can be summarised in a single equation:

$$\partial F = -\mu_0 J\,. \tag{6.86}$$

6.6 Alternative method for the Maxwell equations

In analogy with what is done in geometric algebra we consider a spacetime split of the vector derivative, see chapter 7 of [32] . In an AEST the spacetime split reads

$$\mathrm{e}_4 \partial = \mathrm{e}_4 \left(\mathrm{e}_k \partial_k + \mathrm{e}_4 \partial_4\right) = -\boldsymbol{\sigma}_k \partial_k + \partial_4\,, \tag{6.87}$$

where the bivector $\boldsymbol{\sigma}_k = \mathrm{e}_k \mathrm{e}_4$ as before. The equation

$$\partial F = -\mu_0 J_k \mathrm{e}_k - \mu_0 J_4 \mathrm{e}_4 \tag{6.88}$$

implies

$$\mathrm{e}_4 \partial F = -\mu_0 J_k \mathrm{e}_4 \mathrm{e}_k - \mu_0 J_4 = \mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4\,. \tag{6.89}$$

Substitution of $\mathrm{e}_4 \partial = -\boldsymbol{\sigma}_m \partial_m + \partial_4$ and $F = \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k$ gives

$$\Big(-\boldsymbol{\sigma}_m \partial_m + \partial_4\Big) \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k = \mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4\,. \tag{6.90}$$

The latter can be elaborated to

$$\partial_n \frac{E_n}{c} - I_4 \partial_n B_n + \partial_4 \frac{E_k}{c} \boldsymbol{\sigma}_k - \partial_4 I_4 B_k \boldsymbol{\sigma}_k - \varepsilon_{ijk} \partial_i \frac{E_j}{c} I_4 \boldsymbol{\sigma}_k + \varepsilon_{ijk} \partial_i B_j \boldsymbol{\sigma}_k = \mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4\,. \tag{6.91}$$

Grouping the coefficients leads for the scalar part to

$$\partial_n \frac{E_n}{c} = -\mu_0 J_4 \,, \tag{6.92}$$

for the bivector $\boldsymbol{\sigma}_k$ part to

$$\partial_4 \frac{E_k}{c} + \varepsilon_{ijk}\partial_i B_j = \mu_0 J_k \,, \tag{6.93}$$

for the bivector $I_4\boldsymbol{\sigma}_k$ part to

$$\varepsilon_{ijk}\partial_i \frac{E_j}{c} + \partial_4 B_k = 0 \tag{6.94}$$

and for the quadrivector I_4 part to

$$\partial_n B_n = 0 \,. \tag{6.95}$$

In vector notation they read

$$\nabla \cdot \vec{E} = -\frac{s\rho}{\varepsilon_0 \gamma(u)} \,, \quad \nabla \times \vec{B} + \partial_t \frac{\vec{E}}{c^2} = \mu_0 \vec{j} \,, \quad \nabla \cdot \vec{B} = 0 \,, \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0 \,. \tag{6.96}$$

In a similar way does the equation

$$\partial F \mathrm{e}_4 = -\mu_0 J_k \mathrm{e}_k \mathrm{e}_4 - \mu_0 J_4 = -\mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.97}$$

lead to

$$\nabla \cdot \vec{E} = \frac{s\rho}{\varepsilon_0 \gamma(u)} \,, \quad \nabla \times \vec{B} + \partial_t \frac{\vec{E}}{c^2} = \mu_0 \vec{j} \,, \quad \nabla \cdot \vec{B} = 0 \,, \quad \nabla \times \vec{E} - \partial_t \vec{B} = 0 \,. \tag{6.98}$$

If we suppose there holds

$$\partial \mathrm{e}_4 F = \mu_0 J_k \mathrm{e}_4 \mathrm{e}_k - \mu_0 J_4 = -\mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,, \tag{6.99}$$

then we arrive at

$$\nabla \cdot \vec{E} = \frac{s\rho}{\varepsilon_0 \gamma(u)} \,, \quad \nabla \times \vec{B} - \partial_t \frac{\vec{E}}{c^2} = \mu_0 \vec{j} \,, \quad \nabla \cdot \vec{B} = 0 \,, \quad \nabla \times \vec{E} - \partial_t \vec{B} = 0 \,. \tag{6.100}$$

The conclusion is that none of all the attempts is free of one or more wrong signs in the Maxwell equations.

6.7 Second alternative method for the Maxwell equations

In analogy with what is done in geometric algebra we consider a spacetime split of the vector derivative, see chapter 7 of [32]. In an AEST the spacetime split reads

$$\partial \mathrm{e}_4 = (\mathrm{e}_k \partial_k + \mathrm{e}_4 \partial_4)\, \mathrm{e}_4 = \boldsymbol{\sigma}_k \partial_k + \partial_4 \,, \tag{6.101}$$

where the bivector $\boldsymbol{\sigma}_k = \mathrm{e}_k \mathrm{e}_4$ as before. Suppose there holds

$$\partial \mathrm{e}_4 F = \mu_0 J_k \mathrm{e}_4 \mathrm{e}_k - \mu_0 J_4 = -\mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.102}$$

Then the substitution of $\partial \mathrm{e}_4 = \boldsymbol{\sigma}_m \partial_m + \partial_4$ and $F = \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k$ gives

$$\left(\boldsymbol{\sigma}_m \partial_m + \partial_4\right) \left(\frac{E_k}{c} - I_4 B_k\right) \boldsymbol{\sigma}_k = \mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.103}$$

Performing the algebra we obtain

$$-\partial_n \frac{E_n}{c} + I_4 \partial_n B_n + \partial_4 \frac{E_k}{c} \boldsymbol{\sigma}_k - \partial_4 I_4 B_k \boldsymbol{\sigma}_k + \varepsilon_{ijk} \partial_i \frac{E_j}{c} I_4 \boldsymbol{\sigma}_k - \varepsilon_{ijk} \partial_i B_j \boldsymbol{\sigma}_k = \mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.104}$$

Grouping the coefficients leads to

$$\partial_n \frac{E_n}{c} = \mu_0 J_4 \tag{6.105}$$

for the scalar part,

$$\partial_4 \frac{E_k}{c} - \varepsilon_{ijk} \partial_i B_j = \mu_0 J_k \tag{6.106}$$

for the bivector $\boldsymbol{\sigma}_k$ part,

$$\varepsilon_{ijk} \partial_i \frac{E_j}{c} - \partial_4 B_k = 0 \tag{6.107}$$

for the bivector $I_4 \boldsymbol{\sigma}_k$ part, and

$$\partial_n B_n = 0 \tag{6.108}$$

for the quadrivector I_4 part. In vector notation they read

$$\nabla \cdot \vec{E} = \frac{s\rho}{\varepsilon_0 \gamma(u)} \,, \tag{6.109}$$

$$\nabla \times \vec{B} - \partial_t \frac{\vec{E}}{c^2} = \mu_0 \vec{j} \,, \tag{6.110}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.111}$$

and

$$\nabla \times \vec{E} - \partial_t \vec{B} = 0 \,. \tag{6.112}$$

We see some wrong signs have disappeared at cost of a wrong sign showing up in the latter equation.

6.8 Third alternative method for the Maxwell equations

The equation

$$\partial F = -\mu_0 J_k \mathrm{e}_k - \mu_0 J_4 \mathrm{e}_4 \tag{6.113}$$

implies

$$\partial F \mathrm{e}_4 = -\mu_0 J_k \mathrm{e}_k \mathrm{e}_4 - \mu_0 J_4 = -\mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.114}$$

Substitution of $\partial = \mathrm{e}_m \partial_m + \mathrm{e}_4 \partial_4$ and $F \mathrm{e}_4 = \left(\frac{E_k}{c} - I_4 B_k\right) \mathrm{e}_k$ gives

$$\left(-\mathrm{e}_m \partial_m + \mathrm{e}_4 \partial_4\right) \left(\frac{E_k}{c} - I_4 B_k\right) \mathrm{e}_k = -\mu_0 J_k \boldsymbol{\sigma}_k - \mu_0 J_4 \,. \tag{6.115}$$

Performing the algebra we obtain

$$-\partial_n\frac{E_n}{c}+I_4\partial_n B_n-\partial_4\frac{E_k}{c}\boldsymbol{\sigma}_k-\partial_4 I_4 B_k\boldsymbol{\sigma}_k+\varepsilon_{ijk}\partial_i\frac{E_j}{c}I_4\boldsymbol{\sigma}_k-\varepsilon_{ijk}\partial_i B_j\boldsymbol{\sigma}_k=-\mu_0 J_k\boldsymbol{\sigma}_k-\mu_0 J_4\,. \tag{6.116}$$

Grouping the coefficients leads for the scalar part to

$$\partial_n\frac{E_n}{c}=\mu_0 J_4\,, \tag{6.117}$$

for the bivector $\boldsymbol{\sigma}_k$ part to

$$\partial_4\frac{E_k}{c}+\varepsilon_{ijk}\partial_i B_j=\mu_0 J_k\,, \tag{6.118}$$

for the bivector $I_4\boldsymbol{\sigma}_k$ part to

$$\varepsilon_{ijk}\partial_i\frac{E_j}{c}-\partial_4 B_k=0 \tag{6.119}$$

and for the quadrivector I_4 part to

$$\partial_n B_n=0\,. \tag{6.120}$$

In vector notation they read

$$\nabla\cdot\vec{E}=\frac{s\rho}{\varepsilon_0\gamma(u)}\,, \tag{6.121}$$

$$\nabla\times\vec{B}+\partial_t\frac{\vec{E}}{c^2}=\mu_0\vec{j}\,, \tag{6.122}$$

$$\nabla\cdot\vec{B}=0 \tag{6.123}$$

and

$$\nabla\times\vec{E}-\partial_t\vec{B}=0\,. \tag{6.124}$$

Also here the result is identical to the ones derived before.

Chapter 7

Additional considerations

7.1 Intrinsic redshift

For this section we consider equation (4.64). For an object we will write it as

$$\dot{\tau} = \sqrt{\mathrm{e}^{-2\mu/r} - \mathrm{e}^{2\mu/r}\frac{v^2}{c^2}}\,, \tag{7.1}$$

where v is the velocity of the object. For a photon equation (4.64) is reduced to

$$c_{local} = \mathrm{e}^{-2\mu/r}c\,, \tag{7.2}$$

where we have written the speed of the photon in the gravitational field as c_{local} instead of v in order to avoid confusion with the speed v of the object which emits the photon. For the AEST we assume that emitted frequencies are proportional to the proper time velocity of the emitter. We will denote it as an *intrinsic* redshift. Let f_0 and λ_0 be the frequency and wavelength emitted by an emitter at rest in the absence of gravitation and let f and λ be the corresponding emitted frequency and wavelength in the presence of gravitation. The proper time velocity of an emitter at rest at a distance r from a spherical gravitational source is obtained by substituting $v = 0$ in equation (7.1):

$$\dot{\tau} = \mathrm{e}^{-\mu/r}\,. \tag{7.3}$$

For the frequency of the emitted photon we then have

$$f = \dot{\tau} f_0 = \mathrm{e}^{-\mu/r} f_0\,. \tag{7.4}$$

For the wavelength we obtain

$$\lambda = \frac{c_{local}}{f} = \frac{\mathrm{e}^{-2\mu/r}c}{\mathrm{e}^{-\mu/r}f_0} = \mathrm{e}^{-\mu/r}\frac{c}{f_0} = \mathrm{e}^{-\mu/r}\lambda_0\,. \tag{7.5}$$

That is, the photon moves off with a gravitational blue shift. When the emitted photon propagates out of the gravitational field, its frequency will not change. What changes is the speed of light and the wavelength. When the photon has travelled to an observer far away of the gravitational source, such that μ/r is practically zero, then the velocity of the photon is c and the observed wavelength becomes

$$\lambda_{obs} = \frac{c}{f} = \mathrm{e}^{\mu/r}\lambda_0 \,. \tag{7.6}$$

where r is the distance of the photon with respect to the source mass when it is emitted. For such a photon one will observe a gravitational redshift despite the fact that it has moved off with a gravitational blue shift.

Next we consider an emitter in the absence of gravitation. Then the equation (7.1) reads

$$\dot{\tau} = \sqrt{1 - \frac{v^2}{c^2}} \,, \tag{7.7}$$

For the emitted frequency we then have

$$f = \dot{\tau} f_0 = f_0 \sqrt{1 - \frac{v^2}{c^2}} \,, \tag{7.8}$$

where v is the spatial velocity of the emitter. If there would only be an intrinsic redshift, we would obtain for the wavelength

$$\lambda = \frac{c}{f} = \frac{c}{f_0} \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma(v)\lambda_0 \,. \tag{7.9}$$

Of course, we also have to take into account the Doppler effect due to the motion of the emitter. For a receding emitter the wavelength will be red shifted because of the Doppler effect. The classical Doppler effect reads

$$\lambda(v) = \lambda \left(1 + \frac{v}{c}\right) \,. \tag{7.10}$$

Taking both the intrinsic redshift and the Doppler effect into account, we obtain for the wavelength

$$\lambda(v) = \gamma(v)\lambda_0 \left(1 + \frac{v}{c}\right) = \lambda_0 \sqrt{\frac{c+v}{c-v}} \,. \tag{7.11}$$

The latter is identical to the relativistic Doppler shift.

Finally we consider a receding emitter in the vicinity of a spherical source mass. Then it follows from equation (4.64) for the proper time velocity of the emitter

$$\dot{\tau} = \sqrt{\mathrm{e}^{-2\mu/r} - \mathrm{e}^{2\mu/r}\frac{v^2}{c^2}} = \mathrm{e}^{-\mu/r}\sqrt{1 - \frac{v^2}{c^2_{local}}} \,. \tag{7.12}$$

Without a Doppler effect the emitted frequency and wavelength would be

$$f = f_0\dot{\tau} = f_0 \mathrm{e}^{-\mu/r}\sqrt{1 - \frac{v^2}{c^2_{local}}} \,. \tag{7.13}$$

and

$$\lambda = \frac{c_{local}}{f} = \frac{e^{-2\mu/r}c}{e^{-\mu/r}f_0\sqrt{1-\frac{v^2}{c_{local}^2}}} = \frac{e^{-\mu/r}\lambda_0}{\sqrt{1-\frac{v^2}{c_{local}^2}}} \tag{7.14}$$

respectively. With c_{local} as the local speed of light, the classical Doppler effect reads

$$\lambda(v) = \lambda\left(1+\frac{v}{c_{local}}\right). \tag{7.15}$$

Taking both the intrinsic redshift and the Doppler effect into account, we obtain for the wavelength

$$\lambda(v) = \frac{e^{-\mu/r}\lambda_0}{\sqrt{1-\frac{v^2}{c_{local}^2}}}\left(1+\frac{v}{c_{local}}\right) = e^{-\mu/r}\lambda_0\sqrt{\frac{c_{local}+v}{c_{local}-v}}. \tag{7.16}$$

So, taking into account the Doppler effect, the frequency is

$$f(v) = \frac{c_{local}}{\lambda(v)} = \frac{e^{-\mu/r}c}{\lambda_0}\sqrt{\frac{c_{local}-v}{c_{local}+v}} = e^{-\mu/r}f_0\sqrt{\frac{c_{local}-v}{c_{local}+v}}. \tag{7.17}$$

As mentioned before, the frequency of radiation does not change when the photon moves out of the gravitational field. What increases is the velocity of light and, as a consequence, the wavelength. When the photon has travelled to a position far away of the gravitational source, such that μ/r is practically zero, then the velocity of the photon is c and the observed wavelength becomes

$$\lambda_{obs} = \frac{c}{f(v)} = e^{\mu/r}\lambda_0\sqrt{\frac{c_{local}+v}{c_{local}-v}}. \tag{7.18}$$

When the emitter is far from the gravitational source, $r >> \mu$, then

$$\lambda_{obs} \approx \lambda_0\sqrt{\frac{c+v}{c-v}}. \tag{7.19}$$

However, if the emitter is, for instance, receding with a velocity $v = 0.1c$ at a distance from the gravitational source of, say, $r = 100\mu$, then

$$\lambda_{obs} = e^{0.01}\lambda_0\sqrt{\frac{e^{-0.02}c+0.1c}{e^{-0.02}c-0.1c}} \approx 1.119\lambda_0\,. \tag{7.20}$$

That is, for this example the observed z value for the red shift is

$$z = \frac{\lambda_{obs}-\lambda_0}{\lambda_0} \approx 0.119\,. \tag{7.21}$$

If the observed value $z \approx 0.119$ is interpreted as originating from a free emitter, it will lead to an overestimation of the receding velocity of the emitter: $v \approx 0.112c$ instead of $0.1c$. It may, for instance, lead to a slight overestimation of the Hubble constant.

7.2 Black holes

In the GR the coefficient $g_{tt} = g_{rr}^{-1}$ is, in the Newtonian limit, identified as $1 - 2\phi$, where $\phi = GM/c^2r$ is the gravitational potential. The coefficient $g_{tt} = 1 - 2GM/c^2r$ becomes zero if r is equal to the Schwarzschild radius: $r = 2GM/c^2$. However, for the connection with Newtonian gravitation it is sufficient to identify the coefficient $g_{00} = g_{rr}^{-1}$ to first order as $1-2\phi$. That is, $1-2\phi+\mathcal{O}(\phi^2)$ does the job as well. As argued, a natural choice would be $\mathrm{e}^{-2\phi}$. After rearrangement to the situation in the AEST we then obtained equation (4.44) and for the Lagrangian for dynamics in a gravitational field in an AEST we obtained equation (4.45). Since the factor $\mathrm{e}^{-2\phi}$ never becomes zero, there would be no singularity if one had chosen for the exponential metric. For gravitation in the AEST we solely work with the exponential form. Moreover, the coefficients with the exponential form are not coefficients of the metric. It are just coefficients in the Lagrangian. The metric in an AEST is always $(+,+,+,+)$. This implies that in an AEST there do not exist black holes in the sense that singularities occur if the radius of a dense object is smaller than the Schwarzschild radius. In an AEST there do not exist black holes in the sense that the role of time, parameter time in relativity theory, and the role of space are exchanged for radii smaller than the Schwarzschild radius. However, the AEST does not exclude the possibility of extremely dense masses with radii smaller than the Schwarzschild radius. The radiation of dense source masses will be highly redshifted. It can be imagined that extremely dense masses do not radiate at all. It can also be imagined that the merging of two rotating dense masses, will generate gravitational waves as we observe them on the Earth. However, the explanation of the detection of gravitational waves by means of laser interferometry is different. In an AEST the gravitational waves solely alter the speed of light. They do not alternately contract and expand space. Space is flat and Euclidean in an AEST.

7.3 Hubble's law

In the previous chapters we have given arguments for a spacetime with a Euclidean metric and everywhere flat. As a next step it seems natural to consider spacetime as infinite. That is, we consider the big bang as an expansion in the infinite spacetime and not as the expansion of spacetime itself. Since everything moves at the speed of light, the boundary of the big bang are determined by the objects which move with spatial speed c. For a big bang that happened about $14 \cdot 10^9$ years ago the boundary is a sphere with radius $R \approx 4.3 \cdot 10^3$ Mpc. Objects that have moved with a speed v smaller than c, will have reached a distance D smaller than R. The ratio D/R will be proportional with v/c (This is not completely true since the local velocity of light is smaller than c at the very beginning of the big bang. However, the consequences are negligible).

$$v = \frac{cD}{R}\,. \tag{7.22}$$

It is usually written as

$$v = H_0 D\,, \tag{7.23}$$

where $H_0 \approx 7 \cdot 10^4$ m/s/Mpc is the Hubble constant. Clearly, H_0 is not a constant at all, since it depends on time. Writing c/R as $1/T$ where T is the age of the universe, we have $H_0 = 1/T$ and

$$v = \frac{D}{T}\,. \tag{7.24}$$

For instance, at half the age of the universe the Hubble constant was twice as large as it is now. Ignoring the time dependency of the Hubble constant may lead to an overestimation of the Hubble constant [44] .

7.4 Parameter time

In relativity theory the time t of an observer is regarded as a fourth coordinate of an object. The latter holds for all the observed objects. As a consequence, simultaneous positions of objects always lie on a horizontal line in the Minkowski diagram. On first sight it might seem natural that simultaneous positions will lie on a horizontal line in the Minkowski diagram. However, simultaneity is indissolubly connected with the time ordering of events. It therefore should not serve as a fourth coordinate. In relativity theory the proper time τ of an object is used as its parameter. This is awkward since a parameter time should determine the time ordering of events, while in relativity theory the time ordering is already determined by the observers clock t. The parameterisation problem in relativity theory becomes more apparent when more than one object is observed by the observer. All the objects will have identical fourth coordinates, t, at each instant of time t, while they have different parameters $\tau_1, \tau_2, ...$ for the objects 1, 2, ... All the different proper times cannot be used for an unambiguous determination of simultaneity. According to relativity theory there is no problem since simultaneity is determined by the observers clock t. However, it implies a double role for t as a parameter for the time ordering of events and t as a simultaneous fourth coordinate. It leaves $\tau_1, \tau_2, ...$ as dummy parameters. In the theory of relativity there are problems with the parameterisation of relativistic multibody dynamics and relativistic quantum theories. To deal with the problems one constructs an additional evolution parameter, usually based on the multibody Hamilton equations [36–43] . The additional evolution parameter is usually denoted as τ to distinguish it from the particle proper times $\tau_1, \tau_2, ...$ Although the results obtained with the constructed additional evolution parameter are important and physically meaningful, we are still left with the situation that an evolution parameter τ lives next to the time ordering parameter t. In the theory of relativity the situation for a single particle is $(\vec{x}(\tau), t(\tau))$, with τ the proper time of the particle, and the situation for a multi-particle system is $(\vec{x}_1(\tau), \vec{x}_2(\tau), ..., \vec{x}_n(\tau), t(\tau))$ with τ an additional evolution parameter. While in the AEST theory the situation for a single particle is $(\vec{x}(t), \tau(t))$, with τ the proper time of the particle, and the situation for a multi-particle system is $(\vec{x}_1(t), \tau_1(t), \vec{x}_2(t), \tau_2(t), ..., \vec{x}_n(t), \tau_n(t))$ with τ_k the proper time of particle k. That is, in an AEST there is a single parameter t which is the evolution parameter and the time ordering parameter and there are n fourth coordinates, given by the n proper times, which may differ from each other at each instant of parameter time t. The parameter time t is the proper time of an observer, preferably by an observer at rest in the preferred frame.

7.5 Arrow of time

According to the Feynman-Stueckelberg interpretation an antiparticle does run backwards in time. That is, it runs backwards with respect to the time ordering parameter t as given by the clock of an observer, preferably by an observer at rest in the preferred frame. This is harmless or even useful as long as it is used to calculate the highly accurate results of quantum theory. Nevertheless, it is contra-intuitive or even completely wrong since it implies an antiparticle to flow against the arrow of time. An antiparticle flowing backwards in universal time can not be visualised or imagined. Alternatively, the Feynman-Stueckelberg interpretation is acceptable only if quantum (field) theory is viewed as a calculation model. It should not be seen as a description of true physics. The parameter time t as indicated by the clock at rest in the preferred frame, is the time which determines the order of events and the possible simultaneity of events. When we speak about the arrow of time, we actually think of the arrow of parameter time t. And the sign of this parameter time t will never be reversed. It always ticks forward. However, for the proper times of particles and antiparticles it is quite natural to have opposite signs. The sign of proper time is an individual property of a particle. A negative proper time of an antiparticle just means that its proper time (its clock) runs backwards while the the parameter time t runs forwards. A negative proper time τ has nothing to do with the arrow of time of the whole universe. For an observer which moves with respect to the preferred rest frame and whose proper time is used as parameter time, and whose clocks at the ends of his rods are synchronised (as in relativity theory), simultaneity and the order of events may become reversed. However, this time reversal is not a property of physics or spacetime, it is just an artificial consequence of clock synchronisation. It causes an artificial relativity with a possibility of a reversed causality. Without the clock synchronisation the relativity and the reversed causality disappears. The arrow of time always directs into the future.

Chapter 8

Discussion and other literature

In chapter 1 arguments have been given for an alternative concept of time where the proper time of an object is its the fourth coordinate. The new concept of time leads to an absolute Euclidean spacetime (AEST), where 'absolute' stands for the presence of a preferred frame. In chapter 2 it is shown that experiments which support special relativity can be explained as well in an AEST without relying on an aether theory. In chapter 3 kinematics in an AEST is considered. The AEST action principle is shown to be more in agreement with Snell's law. It is shown that the AEST theory does the same predictions for the Compton effect and pair annihilation as relativity theory. In chapter 4 it is argued that gravitation should be isotropic and exponential. For gravitational lensing, perihelion precession and orbital precession around a bipole mass the AEST Lagrangian for gravitation leads to the same predictions as general relativity. Despite this success it is still a little unsatisfactory that the AEST Lagrangian is an ad hoc Lagrangian. Of course, it would be preferable if it could be derived from first principles. In chapter 5 a tensor analysis of electrodynamics in an AEST is considered. It leads to an alternative formulation of the Lorentz force, Coulomb's law, the Bohr atom and the Maxwell equations. In particular the AEST version of the inhomogeneous Maxwell equations has some sign issues. In chapter 6 various alternative derivations are explored for the Lorentz force and the maxwell equations. Although an interesting exercise, it does not remove the sign issues in the AEST version of the inhomogeneous Maxwell equations. So, the AEST theory is not free of problems. Next to some problems the AEST theory also may have its advantages. For example, the mass ascribed to a photon in an AEST might contribute to a dark matter explanation. As another example, gravitation in an AEST might open a door to the formulation of quantum gravity.

Even if the chapters 2 through 6 are for some reason wrong or unacceptable, it will not take away the problem with Minkowski spacetime: fourth coordinates of different objects having identical values at each instant of parameter time and even equal to parameter time. That is, in Minkowski spacetime the fourth coordinates are just a manifestation of parameter time. In a proper coordinate system the fourth coordinates of different objects must have the freedom to take on values different from each other and different from parameter time. The error is due to our habit to ascribe a time coordinate with our clock and space coordinates with our yardsticks. However, it is more natural to read of the time coordinate of an object from a (virtual) clock attached on that object. An astronaut in a spaceship reads of his x, y, z position from his dashboard. He also reads of time from the clock on his dashboard. in this way, the astronaut obtains its own spacetime coordinates. Of course, the dashboard clock shows the astronauts proper time. To parameterise the successive (x, y, z, τ) coordinates of the spaceship, we need a paramater time t from an external clock. For this, the time t as indicated by a clock of an observer will do: $(x(t), y(t), z(t), \tau(t))$. It differs fundamentally from Minkowski's $(x(\tau), y(\tau), z(\tau), t(\tau))$ coordinate system in special relativity.

From a mathematical point of view a Euclidean space can either be absolute (with a preferred frame) or relative (without a preferred frame). Although an absolute Euclidean spacetime is advocated in the present work, it seems worthwhile to mention an attempt for an relative Euclidean spacetime, where the relativistic length contraction is explained by means of a projection onto the local axis system [44] .

The citations and references in the present book are scarce. Because of the focus on the clarity of explanation an extensive overview of historical or otherwise related papers is absent. Some relevant historical papers can be find in the bibliography of the cited papers. Next to historical papers there is a number of modern papers related to Euclidean space time. In particular the work of Almeida, Brannen and Fontana should be mentioned. For references to their work and for other references, I wish to refer to the overview on the site of Van Linden [45] .

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